Detection of Degenerate Points on the Surface.

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Abstract—**Landslides, bifurcations, multi-saddles and remnants of terraces are distinctive landforms. Some points on the surfaces of these objects are degenerate points. This may help us with their automatic recognition and identification. All first-order and second-order partial derivatives of analyzed function are necessary for detection of degenerate points. Terrain slope, curvatures and Hessian are required for classification of degenerate points. The paper is aimed at detection of fossil landslides. A point of landslide surface where the concave section of thalweg is turning into convex section of ridge line is a degenerate point. Two zero isolines of Hessian and zero isoline of profile, streamline and plan or tangential curvatures pass through this point. Final result of the detection procedure depends to a great extent on the quality of DEM and accuracy of derivatives.**

I. Introduction

Topographic surface may be expressed by the function of two variables x, y in Carthesian coordinates system $\langle 0, x, y, z \rangle$. Let the general formula $z = f(x, y)$ represent a continuously differentiable real function whose second-order partial derivatives exist. The Hessian matrix H of the function $f(x, y)$ is a matrix of second partial derivatives

$$H(x, y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}. \quad (1)$$

Define $D(x, y)$ to be determinant

$$D(x, y) = \det(H(x, y)) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \quad (2)$$

so-called Hessian. Hessian form is identical to the numerator of discriminant of second fundamental form.

Eigenvalues λ_1, λ_2 of the Hessian matrix are computed by solving the quadratic

$$\lambda^2 - \lambda \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + D(x, y) = 0, \quad (3)$$

then

$$\pm\lambda = \frac{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \pm \sqrt{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)^2 - 4D(x, y)}}{2}. \quad (4)$$

If $D(x, y)$ at any point (x_0, y_0) is positive, then osculating paraboloid at point (x_0, y_0) has the form of elliptic paraboloid. If eigenvalues λ_1, λ_2 are positive, the elliptic paraboloid is concave up and if point (x_0, y_0) is a critical point, the function $f(x, y)$ has a local minimum value there. Critical point is the point of function $f(x, y)$ where the gradient vector vanishes, for example if the first partial derivatives are equal to zero. If eigenvalues λ_1, λ_2 are negative, the elliptic paraboloid is concave down and if point (x_0, y_0) is the critical point, the function $f(x, y)$ has a local maximum value there. For elliptical points Dupin indicatrix will form an ellipse aligned with the principal directions. If $D(x, y)$ is negative at the point (x_0, y_0), then osculating paraboloid has the form of hyperbolic paraboloid. If hyperbolic point (x_0, y_0) is the critical point and eigenvalues λ_1, λ_2 have opposite signs, the function $f(x, y)$ has a saddle point there. For hyperbolic point Dupin indicatrix will form a hyperbola. The directions of its asymptotes are the same as asymptotic directions. If $D(x, y) = 0$, then osculating paraboloid has the form of parabolic cylinder and Dupin indicatrix in the point has the form of two parallel lines [3].

Zero isolines of Hessian and streamline curvature together with zero isolines of profile curvature pass through some peaks, pits and double saddle points of isoline field of gradient or slope and together with zero isolines of plan or tangential curvature pass through such points of isoline field of aspect as well [2].

Peaks, depression points and double saddle points on a topographic surface are non-degenerate critical points. Let's

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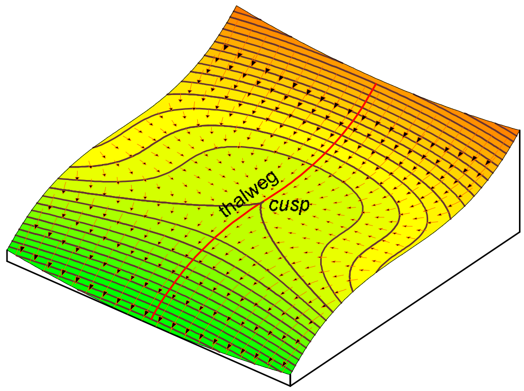

Figure 1. Remnant of terrace surface ($z = x^3 + y^2$)

assume that function $f(x, y)$ or its part is non-Morse function. It means that the Hessian matrix is singular, i.e. Hessian equals to zero at some critical points of function $f(x, y)$. Zero Hessian then defines the degenerate critical points of function $f(x, y)$.

II. Degenerate critical points

Cusp on a remnant of terrace surface in Fig. 1 and central points of double saddle surfaces in Fig. 2 and Fig. 3 and of multi-saddle surfaces [4] are degenerate critical points. Hessian matrix at degenerate critical points have one (e.g. limited cases of saddles or remnants of terraces) or both eigenvalues equal to zero.

Occurrence of degenerate critical points on sufficiently smooth land surface is rare. Such are, for example, multi-saddle points. Additionally, degenerate critical points may be unstable, disappearing even by a small change in altitude. These points appear only for a short time until their disruption (e.g. the limited case of saddle surface in Fig. 2 transforms into various surfaces with double saddle points and limited case of saddle surface, i.e. incipient bifurcation in Fig. 3 often transforms into surface of neighboring valleys with low drainage divide).

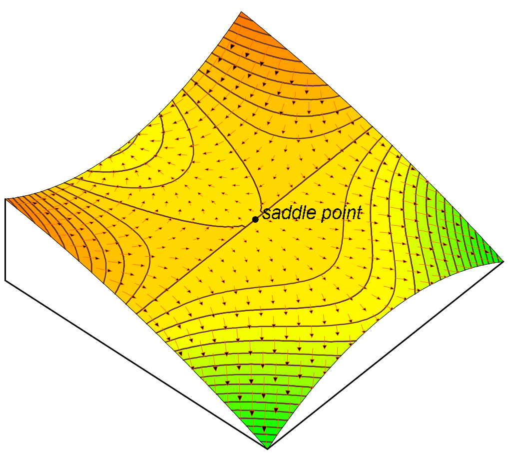

Figure 2. Limited case of saddle surface ($z = y^2 + x^2y$) or ($z = x^2 + y^2x$)

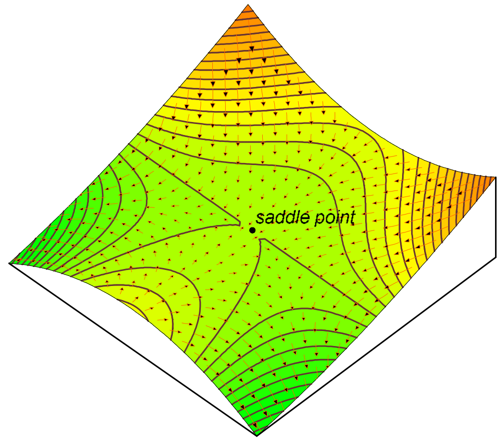

Figure 3. Limited case of saddle surface ($z = y^2 - x^2y$) or ($z = x^2 - y^2x$)

The remnant of terrace surface in Fig. 1 and its inverse surface transform into the surfaces with non-zero gradient magnitude at the central point. Central cusp point vanishes but the point remains an *inflection point* (the inflection point of valley and ridge longitudinal profile, i.e. the inflection point of the thalweg and ridge line). *Inflection point* of transformed surfaces is a regular point (at least one first-order partial derivative is non-zero), though it retains the properties of degenerate critical point (Hessian always equals to zero). We call this point a “degenerate regular” point.

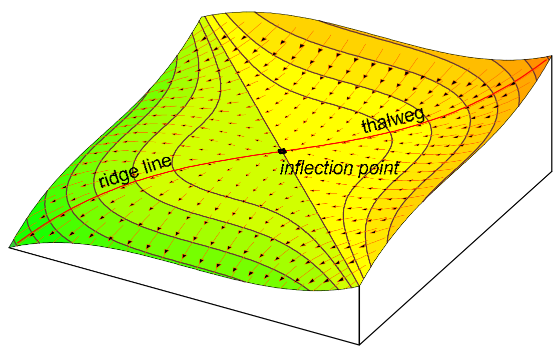

Figure 4. Landslide surface ($z = x^3 \pm y^3$)

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Except for the remnants of terraces, the frequent landforms are landslides. Natural landslide is a dynamic geomorphological form. Sharp edges of an active landslide quickly transform into smooth surface of a fossil landslide. The contours on both sides of central straight contour of ideal landslide surface will bend outward. Typical and degenerated critical point of an ideal landslide surface is the *inflection point* where a concave section of the thalweg is turning into convex section of the ridge line (Fig. 4). All derivatives from the Hessian matrix, and thus also both zero eigenvalues at *inflection point* of ideal landslide surface equal to zero.

III. Detection of Degenerate Points

First partial derivatives and zero Hessian define degenerate critical or regular points. Not all points with zero Hessian are important marks on the topographic surface. Important marks are the points mentioned above. In order to determine degenerate points, the derivatives and curvatures have to be applied.

The course of zero isolines of triplet curvatures in the immediate neighborhood of the regular *inflection point* of an ideal landslide surface is illustrated in Fig. 5. Plan or tangential and profile curvatures are commonly used curvatures in current geomorphometry. Additional curvature can be given by

$$A_n = \frac{\partial A}{\partial n} = \frac{\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) + \frac{\partial^2 f}{\partial x \partial y}\left(\left(\frac{\partial f}{\partial y}\right)^2 - \left(\frac{\partial f}{\partial x}\right)^2\right)}{\sqrt{\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)^3}} \qquad (5)$$

when A is an aspect (values 0° and 360° correspond to the south direction) and normal direction n is a direction in physical terms [1]. Shary called the curvature with opposite sign "rotor" [6] and Peckham called it "streamline curvature" [5]. Streamline curvature expresses a curvature of flow lines in the plane (x, y).

Two zero isolines of Hessian and zero isolines of all second-order partial derivatives pass through an *inflection point*, which divides landslide surface into the erosional and depositional landforms. Products of first and second partial derivatives define the numerators of the formulas of all curvatures, and therefore zero isolines of curvatures pass through the *inflection point* of landslide surface from Fig. 5 as well.

In the case of remnant of terrace surface from Fig. 6, the zero isoline of Hessian and zero isoline of profile and streamline curvature pass through a regular *inflection point* of the thalweg. Two zero isolines of plan or tangential curvature in the contours direction only determine the neighborhood of the *inflection point*.

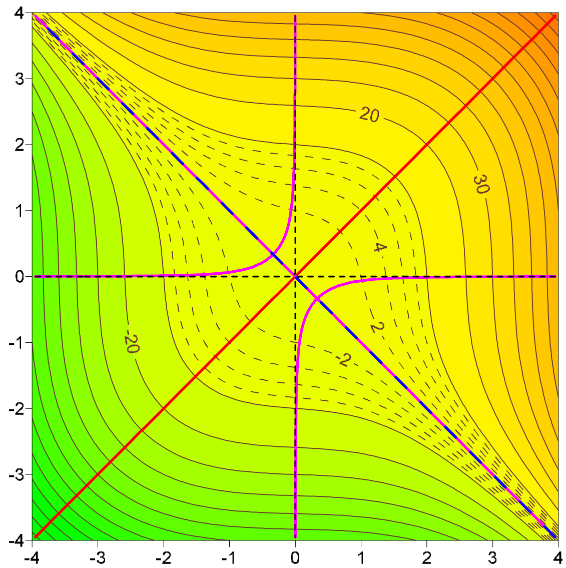

Figure 5. Landslide surface ($z = x^3 + y^3 + x + y$): brown isolines – contours, red isoline – zero streamline curvature (thalweg and ridge line), dashed blue isoline – zero profile curvature, magenta isolines – zero plan or tangential curvature, dashed black isolines – zero Hessian

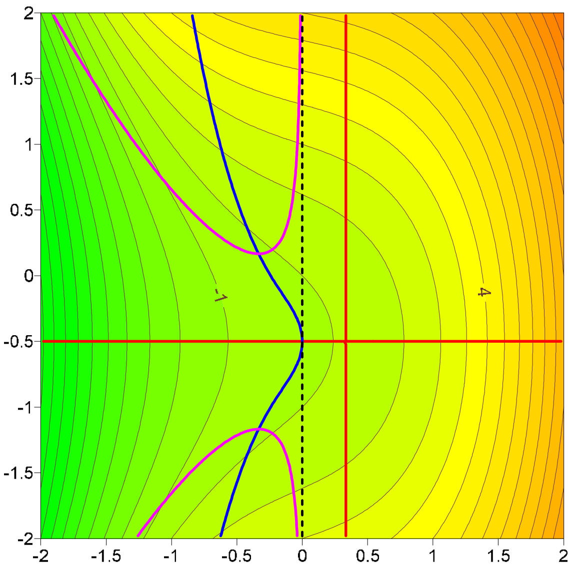

Figure 6. Remnant of terrace surface ($z = x^3 + y^2 + x + y$): brown isolines – contours, red isolines – zero streamline curvature, blue isoline – zero profile curvature, magenta isolines – zero plan or tangential curvature, dashed black isoline – zero Hessian

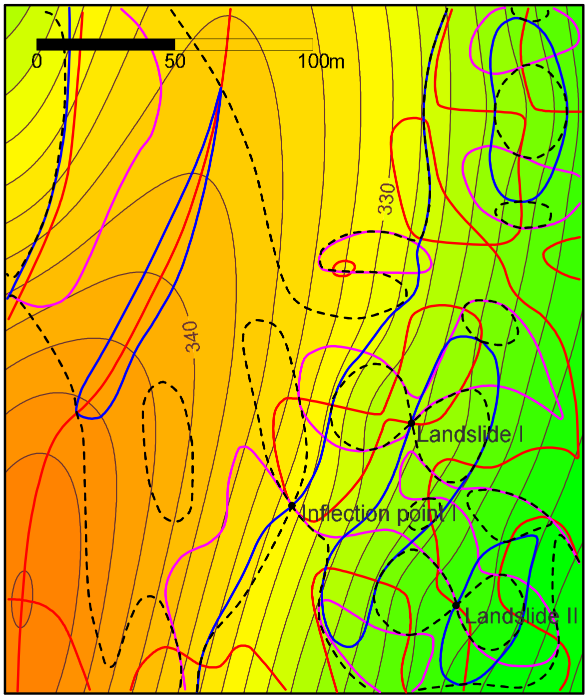

Figure 7. Result of landslide detection procedure: brown isolines – contours, red isolines – zero streamline curvature, blue isolines – zero profile curvature, magenta isolines – zero plan or tangential curvature, dashed black isolines – zero Hessian

Result of the detection procedure is shown in Fig. 7. In the figure, three points may be the *inflection points* of fossil landslides. Two zero isolines of Hessian and zero isolines of streamline, profile and plan/tangential curvatures pass through the *inflection point* of the Landslide I and Landslide II and the Inflection point I. The Landslide I and Landslide II points determine landslide surfaces evident from contours.

The Inflection point I is not a local minimum but a saddle point in the gradient magnitude or slope angle isoline field. This means that profile curvature in the immediate neighborhood of the point is positive (convex) in the up-slope direction and negative (concave) in the down-slope direction. The Inflection point I is not an *inflection point* of landslide surface. The gradient magnitude or slope in the surrounding area of the Inflection point I is greater than the gradient magnitude or slope in the more distant areas where head scarp and toe should be in the case of real landslide.

The intersection points of zero isolines of Hessian and zero isolines of streamline and profile curvature and the intersection points of zero isolines of Hessian and zero isolines of the streamline and plan or tangential curvature in Fig. 7 delimit more or less inclined slopes which resemble remnant parts of various terraces. Such shapes are also landslide scarp and debris bulge.

IV. Conclusions

Two zero isolines of Hessian and zero isoline of profile, streamline and plan or tangential curvature pass through the point where a concave thalweg is turning into a convex ridge line or, on the contrary, a convex thalweg is turning into a concave ridge line. It is similar to the situation when a ridge line is turning into a thalweg. All second-order partial derivatives at the point are equal to zero.

The test of second-order partial derivatives is a principle of a certain procedure to detect degenerate points with two zero eigenvalues. Supplementary conditions needed for differentiation of points can be very simple: for example, sign of profile curvature in the down-slope and up-slope direction or kind of singularity of gradient magnitude or slope angle isoline field. The method for extraction of potential fossil landslide shapes performs better if the surface is sufficiently smooth.

Acknowledgment

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