An extension to the spherical metric using polar linear interpolation

DANIEL J. GREENHOE

Abstract: The *spherical metric* p_r operates on the surface of a *sphere* with radius r centered at the origin in a *linear space* $\mathbb{R}^N$. Thus, for any pair of points (p,q) *on the surface* of this sphere, (p,q) is in the *domain* of p_r and $\mathsf{p}_r(p,q)$ is the "distance" between those points. However, if x and y *are* both in $\mathbb{R}^N$ but are *not* on the surface of a common sphere centered at the origin, then (p,q) is *not* in the domain of p_r and $\mathsf{p}_r(p,q)$ is simply *undefined*. In certain applications, however, it would be useful to have an *extension* d of p to the entire space $\mathbb{R}^N$ (rather than just on a surface in $\mathbb{R}^N$). Real world applications for such an extended metric include calculations involving near earth objects, and for certain distance spaces useful in symbolic sequence processing. This paper introduces an extension to the spherical metric using a polar form of *linear interpolation*. The extension is herein called the *Lagrange arc distance*. It has as its domain the entire space $\mathbb{R}^N$, is *homogeneous*, and is *continuous* everywhere in $\mathbb{R}^N$ except at the origin. However the extension does come at a cost: The *Lagrange arc distance* $\mathsf{d}(p,q)$, as its name suggests, is a *distance function* rather than a *metric*. In particular, the *triangle inequality* does not in general hold. Moreover, it is *not translation invariant*, does *not induce a norm*, and balls in the *distance space* $(\mathbb{R}^N, \mathsf{d})$ are *not convex*. On the other hand, empirical evidence suggests that the *Lagrange arc distance* results in structure similar to that of the *Euclidean metric* in that balls in $\mathbb{R}^2$ and $\mathbb{R}^3$ generated by the two functions are in some regions of $\mathbb{R}^N$ very similar in form.

2010 Mathematics Subject Classification: 54E25 (primary); 54A05,54A20 (secondary)

Universal Decimal Classification (UDC): 517+519.6

Keywords: metrics, spherical metric, metric space, distance space, polar coordinates, interpolation, Lagrange interpolation

Contents

1 Standard definitions

1.1 Sets

Definition 1.1 [1] Let X be a *set*.[2] The **empty set** $\varnothing$ is defined as $\varnothing \triangleq \{x \in X | x \neq x\}$.

Definition 1.2 [3] Let $\mathbb{R}$ be the **set of real numbers**. Let $\mathbb{R}^{\vdash} \triangleq \{x \in \mathbb{R} | x \geq 0\}$ be the **set of non-negative real numbers**. Let $\mathbb{R}^{+} \triangleq \{x \in \mathbb{R} | x > 0\}$ be the **set of postive real numbers**. Let $\mathbb{Z}$ be the **set of integers**. Let $\mathbb{W} \triangleq \{n \in \mathbb{Z} | n \geq 0\}$ be the **set of whole numbers**. Let $\mathbb{N} \triangleq \{n \in \mathbb{Z} | n \geq 1\}$ be the **set of natural numbers**. Let $\mathbb{Z}^{*} \triangleq \mathbb{Z} \cup \{-\infty, \infty\}$ be the *extended set of integers*.

1.2 Relations

One of the most fundamental structures in mathematics is the *ordered pair*, and one of the most common definitions of *ordered pair* is due to *Kuratowski* (1921) and is presented next:[4]

Definition 1.3 [5] The **ordered pair** (a, b) is defined as $(a, b) \triangleq \{\{a\}, \{a, b\}\}$.

Proposition 1.4 (next) and Corollary 1.5 demonstrate that the the definition of *ordered pair* given by Definition 1.3 allows a and b to be unambiguously extracted from (a, b) and that (a, b) is well defined.

Proposition 1.4 *Let $\cap$ be the set intersection operator and $\triangle$ the set symmetric difference operator.*

$$\begin{aligned} \{a\} &= \bigcap(a,b) &= \bigcap\{\{a\},\{a,b\}\} &= \{a\} \cap \{a,b\} \\ \{b\} &= \triangle(a,b) &= \triangle\{\{a\},\{a,b\}\} &= \{a\} \triangle \{a,b\} \end{aligned}$$

Corollary 1.5 [6] $(a, b) = (c, d) \iff \{a = c \text{ and } b = d\}$

[1] Halmos (1960) page 8, Kelley (1955) page 3, Kuratowski (1961), page 26

[2] The mathematical structure called *set* is left undefined in this paper. For more information on *sets*, see for example Zermelo (1908a) pages 263–267 (7 axioms), Zermelo (1908b) ⟨(English translation of previous reference)⟩, Fraenkel (1922), Halmos (1960) pages 1–6 ⟨Naive set theory⟩, Wolf (1998), page 139

[3] Notation $\mathbb{R}, \mathbb{W}, \mathbb{N}$, etc.: *Bourbaki notation*. References: Davis (2005) page 9, Cohn (2012) page 3

[4] As an alternative to the Kuratowski definition, the *ordered pair* can also be taken as an *axiom*. References: Bourbaki (1968), page 72, Munkres (2000), page 13

[5] Suppes (1972) page 32, Halmos (1960) page 23, Kuratowski (1961), page 39, Kuratowski (1921) ⟨Def. V, page 171⟩, Wiener (1914)

[6] Apostol (1975) page 33, Hausdorff (1937) page 15

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

✎PROOF:
$$\begin{array}{lllll} \{a\} &=& \bigcap(a,b) &=& \bigcap(c,d) &=& \{c\} & \text{by Proposition 1.4 and left hypothesis}\\ \{b\} &=& \triangle(a,b) &=& \triangle(c,d) &=& \{d\} & \text{by Proposition 1.4 and left hypothesis}\\ (a,b) &=& (c,d) & & & & & \text{by right hypothesis}\end{array}$$
✏

Definition 1.6 [7] Let X and Y be sets. The **Cartesian product** $X \times Y$ is defined as
$$X \times Y \triangleq \{(x,y)\,|(x \in X) \text{ and } (y \in Y)\}$$

Definition 1.7 [8] Let X and Y be *sets*. A **relation** ® on X and Y is any subset of $X \times Y$ such that $® \subseteq X \times Y$. The set 2^{XY} is the **set of all relations** in $X \times Y$.

Definition 1.8 [9] Let X and Y be *sets*. A *relation* $\mathsf{f} \in 2^{XY}$ is a **function** if
$\{(x,y_1) \in \mathsf{f} \quad \text{and} \quad (x,y_2) \in \mathsf{f}\} \implies \{y_1 = y_2\}$. The set Y^X is the **set of all functions** in 2^{XY}.

Definition 1.9 Let X be a set. The quantity 2^X is the *power set of* X such that
$$2^X \triangleq \{A \subseteq X\} \quad \text{(the set of all subsets of } X\text{)}.$$

Definition 1.10 Let Y be a *set*. The structure Y^n for $n \in \mathbb{N}$ is a *set* defined as
$$\begin{aligned} Y^1 &\triangleq Y && \text{and}\\ Y^n &\triangleq Y \times Y^{n-1} && \text{for } n = 2,3,4,\ldots \end{aligned}$$

Example 1.11 The set $\mathbb{R}^N$ is the *N-dimensional real space*, where $\mathbb{R}$ is the *set of real numbers* (Definition 1.2 page 2).

Definition 1.12 [10] Let $® \in 2^{XY}$ be a *relation* (Definition 1.7 page 3).

The **domain** of ® is $\mathcal{D}(®) \triangleq \{x \in X \,|\, \exists y \text{ such that } (x,y) \in ®\}$.
The **image set** of ® is $\mathcal{I}(®) \triangleq \{y \in Y \,|\, \exists x \text{ such that } (x,y) \in ®\}$.
The **range** of ® is any set $\mathcal{R}(®)$ such that $\mathcal{I}(®) \subseteq \mathcal{R}(®)$.

1.3 Order

Definition 1.13 [11] Let X be a set. A relation $\leq$ is an **order relation** in 2^{XX} (Definition 1.7 page 3) if

1. $x \leq x$ $\quad \forall x \in X$ (*reflexive*) and — *preorder*
2. $x \leq y$ and $y \leq z \implies x \leq z$ $\quad \forall x,y,z \in X$ (*transitive*) and — *preorder*
3. $x \leq y$ and $y \leq x \implies x = y$ $\quad \forall x,y \in X$ (*anti-symmetric*).

An **ordered set** is the pair $(X, \leq)$. If $x \leq y$ or $y \leq x$, then elements x and y are said to be **comparable**; otherwise they are **incomparable**. The relation $<$ is the relation $\leq \setminus =$ ("less than but not equal to"), where $\setminus$ is the *set difference* operator, and $=$ is the equality relation. If every pair of elements in $(X, \leq)$ is *comparable*, then $(X, \leq)$ is said to be **linearly ordered** or **totally ordered**.

Example 1.14 The pair $(\mathbb{R}, \leq)$, where $\mathbb{R}$ is the *set of real numbers* and $\leq$ is the *standard ordering relation on* $\mathbb{R}$, is *linearly ordered/totally ordered*.

[7] Halmos (1960) page 24, G. Frege, 2007 August 25, `http://groups.google.com/group/sci.logic/msg/3b3294f5ac3a76f0`

[8] Maddux (2006) page 4, Halmos (1960) pages 26–30, Suppes (1972) page 86, Kelley (1955) page 10, Bourbaki (1939), Bottazzini (1986) page 7, Comtet (1974) page 4 ⟨$|Y^X|$⟩; The notation 2^{XY} is motivated by the fact that for finite X and Y, $|2^{XY}| = 2^{|X|\cdot|Y|}$.

[9] Maddux (2006) page 4, Halmos (1960) pages 26–30, Suppes (1972) page 86, Kelley (1955) page 10, Bourbaki (1939), Bottazzini (1986) page 7, Comtet (1974) page 4 ⟨$|Y^X|$⟩; The notation Y^X is motivated by the fact that for finite X and Y, $|Y^X| = |Y|^{|X|}$.

[10] Munkres (2000), page 16, Kelley (1955) page 7

[11] MacLane and Birkhoff (1999) page 470, Beran (1985) page 1, Korselt (1894) page 156 ⟨I, II, (1)⟩, Dedekind (1900) page 373 ⟨I–III⟩. An *order relation* is also called a **partial order relation**. An *ordered set* is also called a **partially ordered set** or **poset**.

Definition 1.15 [12] In an *ordered set* $(X, \leq)$,

the *set*	$[x : y]$	$\triangleq$ $\{z \in X \mid x \leq z \leq y\}$	is called a	**closed interval**	on $(X, \leq)$	and
the *set*	$(x : y]$	$\triangleq$ $\{z \in X \mid x < z \leq y\}$	is called a	**half-open interval**	on $(X, \leq)$	and
the *set*	$[x : y)$	$\triangleq$ $\{z \in X \mid x \leq z < y\}$	is called a	**half-open interval**	on $(X, \leq)$	and
the *set*	$(x : y)$	$\triangleq$ $\{z \in X \mid x < z < y\}$	is called an	**open interval**	on $(X, \leq)$	.

2 Introduction

2.1 The spherical metric

The *spherical metric*, or *great circle metric*,[13] p_r operates on the *surface of a sphere with radius r* centered at the origin $(0, 0, \cdots, 0)$ in a linear space $\mathbb{R}^N$, where "*surface of a sphere...*" is defined as all the points in $\mathbb{R}^N$ that are a distance r from $(0, 0, \cdots, 0)$ with respect to the *Euclidean metric* (Definition B.3 page 22). Thus, for any pair of points (p, q) *on the surface* of this sphere, (p, q) is in the *domain* of p_r and $\mathsf{p}_r(p, q)$ is the "distance" between those points. However, if p and q *are* both in $\mathbb{R}^N$ but are *not* on the surface of a common sphere centered at the origin, then (p, q) is *not* in the domain of p_r and $\mathsf{p}_r(p, q)$ is simply *undefined*.

In certain applications, however, it would be useful to have an *extension* d of the spherical metric p to the entire space $\mathbb{R}^N$ (rather than just on a surface in $\mathbb{R}^N$). For example, for the points $p \triangleq (0, 1)$ and $q \triangleq (1, 0)$ (which are both on the surface of a common sphere in $\mathbb{R}^2$), we would like d to be compatible with p such that $\mathsf{d}(p, q) = \mathsf{p}(p, q)$. If $r \triangleq (2, 0)$, then the pair (p, r) is *not* in the domain of p, but we still would like it to be in the domain of d such that $\mathsf{d}(p, r)$ is defined—and in this way d would be an *extension* of p.

Real world applications for such an extended metric include calculations involving natural and man-made objects entering/exiting the earth's atmosphere (Section 4.2 page 21) and for certain distance spaces useful in symbolic sequence processing (Section 4.1 page 18).

2.2 Linear interpolation

This paper introduces an extension to the *spherical metric* based on a polar form of *linear interpolation*. *Interpolation* has a very long history with evidence suggesting that it extends possibly all the way back to the Babylonians living around 300BC.[14]

Linear interpolation between two points $p \triangleq (x_1, y_1)$ and $q \triangleq (x_2, y_2)$ in $\mathbb{R}^2$ is conveniently and intuitively expressed in a *Cartesian coordinate system* using what is commonly known as *Lagrange interpolation* (Definition C.1 page 22) in the form

$$y = y_1\left(\frac{x - x_2}{x_1 - x_2}\right) + y_2\left(\frac{x - x_1}{x_2 - x_1}\right).$$

Newton interpolation (Definition C.4 page 23) yields the same expression, but generally requires more "effort"

[12] Apostol (1975) page 4, Ore (1935) page 409, Duthie (1942) page 2, Ore (1935) page 425 ⟨*quotient structures*⟩

[13] Ratcliffe (2013) pages 37–38 ⟨The Spherical Metric⟩, Deza and Deza (2014) page 123 ⟨6.4 Non-Euclidean Geometry⟩, Deza and Deza (2006) page 73 ⟨6.4 Non-Euclidean Geometry⟩, Silver and Stokes (2007) page 9

[14] Meijering (2002) page 320

(back substitution or matrix algebra):

$$y \triangleq \underbrace{\sum_{k=1}^{2} \alpha_k \sum_{m=1}^{k} (x - x_m)}_{\textit{Newton polynomial (Definition C.4)}} = \alpha_1 \left[x - x_1 \right] + \alpha_2 \left[(x - x_1) + (x - x_2) \right] = (\alpha_1 + \alpha_2)(x - x_1) + \alpha_2 (x - x_2)$$

$$y_1 = \alpha_1 \left[x_1 - x_1 \right] + \alpha_2 \left[(x_1 - x_1) + (x_1 - x_2) \right] \implies \alpha_2 = \frac{y_1}{x_1 - x_2}$$

$$y_2 = \alpha_1 \left[x_2 - x_1 \right] + \alpha_2 \left[(x_2 - x_1) + (x_2 - x_2) \right] \implies \alpha_1 = \frac{y_1 + y_2}{x_2 - x_1}$$

$$y = \left[\frac{y_1 + y_2}{x_2 - x_1} + \frac{y_1}{x_1 - x_2} \right] (x - x_1) + \left[\frac{y_1}{x_1 - x_2} \right] (x - x_2) = y_1 \left(\frac{x - x_2}{x_1 - x_2} \right) + y_2 \left(\frac{x - x_1}{x_2 - x_1} \right)$$

Of course the 2-point *Lagrange interpolation*/ *Newton interpolation* polynomial can also be written in the familiar *slope-intercept* $y = mx + b$ form as

$$y = y_1 \left(\frac{x - x_2}{x_1 - x_2} \right) + y_2 \left(\frac{x - x_1}{x_2 - x_1} \right) = y_2 \left(\frac{x - x_1}{x_2 - x_1} \right) - y_1 \left(\frac{x - x_2}{x_2 - x_1} \right) = \frac{y_2 (x - x_1) - y_1 (x - x_2)}{x_2 - x_1}$$

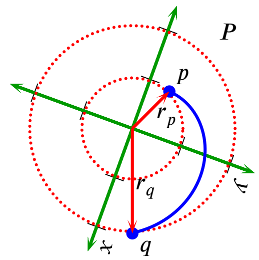
$$= \underbrace{\left(\frac{y_2 - y_1}{x_2 - x_1} \right)}_{\text{slope}} x + \underbrace{\left(\frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} \right)}_{y\text{-intercept}}$$

2.3 Polar linear interpolation

Linear interpolation is often illustrated in terms of cartesian coordinates (x, y). But there is no reason why the same principles cannot be used in terms of polar coordinates $(\mathrm{r}(\theta), \theta)$. However care does need to be taken where θ may be interpreted to “jump” from 2π to 0 or from $-\pi$ to π.

Here is an expression for 2-point *Lagrange interpolation*/ *Newton interpolation* in *polar form*:

$$\mathrm{r}(\theta) \triangleq r_p \left[\frac{\theta - \theta_q}{\theta_p - \theta_q} \right] + r_q \left[\frac{\theta - \theta_p}{\theta_q - \theta_p} \right] \quad \forall \theta \in \left[\theta_p : \theta_q \right]$$



Note the following:

(1) The orientation of the axes in plane P is arbitrary, and that without loss of generality we can orient the axes such that p or q is on the positive x-axis and that the other point has a non-negative y value.

(2) This means that the length of the arc between p at (r_p, θ_p) and q at (r_q, θ_q) under the original orientation is equal to the length of the arc between the points $(r_p, 0)$ and $(r_q, |\theta_p - \theta_q|)$ in the new orientation.

(3) One important reason for the geometrical acrobatics here is that we don't want to have to calculate the values for θ_p and θ_q in a plane P (which we don't even immediately have an algebraic expression for anyways). But calculating the value $\phi \triangleq |\theta_p - \theta_q|$ is quite straightforward because the “*dot product*” $\langle p \,|\, q \rangle$ of p and q (which is very easy to calculate) in $\mathbb{R}^N$ equals $r_p r_q \cos\phi$ (and so $\phi = \arccos\left(\frac{1}{r_p r_q} \langle p \,|\, q \rangle \right)$).

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

(4) Actually, $\phi = |\theta_q - \theta_p|$, as demonstrated below:

$$\begin{aligned}
\phi &\triangleq \arccos\left(\frac{1}{r_p r_q}\sum_{n=1}^{N} x_n y_n\right) && \text{by definition of } \phi \\
&\triangleq \arccos\left(\frac{1}{r_p r_q}\langle p \mid q\rangle\right) && \text{a standard definition from the field of "linear algebra"} \\
&= \arccos\left(\frac{1}{r_p r_q}\left[r_p r_q \cos|\theta_q - \theta_p|\right]\right) && \text{a standard result from the field of "linear algebra"} \\
&= \left\{|\theta_q - \theta_p|, 2\pi - |\theta_q - \theta_p|\right\} && \text{by definition of } \arccos(x) \text{ and } \cos(x) \\
&= |\theta_q - \theta_p| && \text{by item (1)}
\end{aligned}$$

(5) Setting $\theta_p = 0$ and $\theta_q = \phi$ yields the following:

$$\begin{aligned}
\mathsf{r}(\theta) &= r_p\left[\frac{\theta - \theta_q}{\theta_p - \theta_q}\right] + r_q\left[\frac{\theta - \theta_p}{\theta_q - \theta_p}\right] && \text{Langrange form (Definition C.1 page 22)} \\
&= r_p\left[\frac{\theta - \phi}{0 - \phi}\right] + r_q\left[\frac{\theta - 0}{\phi - 0}\right] = \frac{-r_p\theta + r_p\phi + r_q\theta}{\phi} \\
&= \left(\frac{r_q - r_p}{\phi}\right)\theta + r_p && \text{polar slope-intercept form}
\end{aligned}$$

2.4 Distance in terms of polar linear interpolation arcs

This paper introduces a new function herein called, for better or for worse,[15] the *Lagrange arc distance* (Definition 3.1 page 7) $\mathsf{d}(p, q)$. It's domain is the entire space $\mathbb{R}^N$. This function has been found useful in *symbolic sequence processing* (Section 4.1 page 18). It is an extension of the *spherical metric*, which only has as domain the surface of a sphere in $\mathbb{R}^N$.

When p or q is at the origin, or when the polar angle ϕ between p and q is 0, then the *Lagrange arc distance* $\mathsf{d}(p, q)$ is simply a $\frac{1}{\pi}$ scaled *Euclidean metric* (Definition B.3 page 22). In all other cases, $\mathsf{d}(p, q)$ is the $\frac{1}{\pi}$ scaled length of the *Lagrange interpolation* arc extending from p to q.

An equation for the length of an arc in polar coordinates is[16]

$$\mathsf{R}(p, q) = \int_{\theta_p}^{\theta_q} \sqrt{r^2(\theta) + \left(\frac{\mathsf{d}r}{\mathsf{d}\theta}\right)^2}\, \mathsf{d}\theta$$

This integral may look intimidating. Later however, Theorem 3.3 (page 9) demonstrates that it has an "easily" computable and straightforward solution only involving *arithmatic operators* $(+, -, \ldots)$, the *absolute value* function $|x|$, the *square root* function $\sqrt{x}$, and the *natural log* function $\ln(x)$.

Finally, note that the extension does come at a cost—the *Lagrange arc distance* is not a *metric* (Definition B.1 page 22), but rather only a *distance* (Definition A.1 page 21, Theorem 3.13 page 15). For more details about the impact of this cost, see Theorem A.4 (page 21).

[15] "for better or for worse": As already pointed out, *Newton interpolation* or simply the slope-intercept form $y = mx + b$ of the line equation can with a little bit of effort give you the same equation as the 2-point *Lagrange interpolation*. So why not name the function $\mathsf{d}(p, q)$ of Definition 3.1 "Newton arc distance"? Actually Newton published his interpolation method (for example in his 1711 "Methodus differentialis" Newton (1711)) long before Lagrange (Lagrange (1877)). But besides that, Lagrange was not really the first to discover what is commonly called "Lagrange interpolation". The same result was actually published about 98 years earlier by Edward Waring (Waring (1779)). But in the end, the choice to use the name "Lagrange arc distance" has some justification in that it's form arguably comes more readily using Lagrange interpolation than it does from *Newton interpolation* (which requires back substitution); and even though "Lagrange interpolation" probably should be called "Waring interpolation", the fact is that it's normally called "Lagrange interpolation". So there is some motivation for the choice of the name. And "for better or for worse", the function $\mathsf{d}(p, q)$ is herein called the "Lagrange arc distance". …One last note: for a much fuller historical background of interpolation, see Meijering (2002).

[16] Stewart (2012) page 533 ⟨Section 9.4 Areas and lengths in polar coordinates⟩

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

3 Lagrange arc distance

3.1 Definition

Definition 3.1 Let $p \triangleq (x_1, x_2, \cdots, x_N)$ and $q \triangleq (y_1, y_2, \cdots, y_N)$ be two points in the *space* $\mathbb{R}^N$ with origin $(0, 0, \cdots, 0)$. Let

$$\underbrace{r_p \triangleq \left(\sum_{n=1}^{N} x_n^2\right)^{\frac{1}{2}}}_{\text{(magnitude of } p)} \qquad \underbrace{r_q \triangleq \left(\sum_{n=1}^{N} y_n^2\right)^{\frac{1}{2}}}_{\text{(magnitude of } q)} \qquad \underbrace{\phi \triangleq \arccos\left(\frac{1}{r_p r_q}\sum_{n=1}^{N} x_n y_n\right)}_{\text{(angle between } p \text{ and } q)}$$

$$\underbrace{\mathsf{r}(\theta) \triangleq \left(\frac{r_q - r_p}{\phi}\right)\theta + r_p}_{\text{(polar interpolation polynomial)}} \qquad \underbrace{\mathsf{R}(p,q) = \int_0^{\phi} \sqrt{\mathsf{r}^2(\theta) + \left(\frac{\mathsf{d}\mathsf{r}}{\mathsf{d}\theta}\right)^2}\,\mathsf{d}\theta}_{\text{(length of the arc } (\mathsf{r}(\theta),\theta) \text{ between } p \text{ and } q)}$$

The **Langrange arc distance** $\mathsf{d}(p,q)$ is defined as

$$\mathsf{d}(p,q) = \begin{cases} {}^1\!/\!_\pi\left|r_p - r_q\right| & \text{if } p = (0,0,\cdots,0) \text{ or } q = (0,0,\cdots,0) \text{ or } \phi = 0 \\ {}^1\!/\!_\pi \mathsf{R}(p,q) & \text{otherwise} \end{cases} \qquad \forall p,q \in \mathbb{R}^N$$

3.2 Calculation

The integral in Definition 3.1 may look intimidating. However, Theorem 3.3 (page 9) demonstrates that it has an "easily" computable and straightforward solution only involving *arithmatic operators* $(+, -, \ldots)$, the *absolute value* function $|x|$, the *square root* function $\sqrt{x}$, and the *natural log* function $\ln(x)$. But first, a lemma (next) to help with the proof of Theorem 3.3.

Lemma 3.2 [17] *Let $\sqrt{x} \in \mathbb{R}^{\mathbb{R}}$ be the square root function, and $\ln(x) \triangleq \log_e(x) \in \mathbb{R}^{\mathbb{R}}$ be the natural log function. Let ε be any given value in $\mathbb{R}$.*

$$\left\{2ax + b + 2\sqrt{a(ax^2+bx+c)} > 0\right\} \implies$$
$$\left\{\int \sqrt{ax^2+bx+c}\,\mathsf{d}x = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a^{3/2}}\ln\left(2ax+b+2\sqrt{a(ax^2+bx+c)}\right) + \varepsilon\right\}$$

✎PROOF:

(1) lemma:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{2ax+b}{4a}\right)\sqrt{ax^2+bx+c}\right] &= \left(\frac{2ax+b}{4a}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{ax^2+bx+c}\right) + \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{2ax+b}{4a}\right)\left(\sqrt{ax^2+bx+c}\right) && \text{by } \textit{prod. rule} \\
&= \left(\frac{2ax+b}{4a}\right)\left(\frac{2ax+b}{2\sqrt{ax^2+bx+c}}\right) + \left(\frac{2a}{4a}\right)\left(\sqrt{ax^2+bx+c}\right) && \text{by } \textit{chain rule} \\
&= \frac{(2ax+b)^2 + 4a(ax^2+bx+c)}{8a\sqrt{ax^2+bx+c}} \\
&= \frac{4a^2x^2 + 4axb + b^2 + 4a^2x^2 + 4abx + 4ac}{8a\sqrt{ax^2+bx+c}} \\
&= \frac{8a^2x^2 + 8abx + b^2 + 4ac}{8a\sqrt{ax^2+bx+c}}
\end{aligned}$$

[17] Gradshteyn and Ryzhik (2007) page 94 ⟨2.25 Forms containing $\sqrt{a+bx+cx^2}$, 2.26 Forms containing $\sqrt{a+bx+cx^2}$ and integral powers of x⟩ Jeffrey (1995) page 160 ⟨4.3.4 Integrands containing $(a+bx+cx^2)^{1/2}$⟩, Jeffrey and Dai (2008) pages 172–173 ⟨4.3.4 Integrands containing $(a+bx+cx^2)^{1/2}$⟩,

(2) lemma: If $\left(2ax+b+2\sqrt{a(ax^2+bx+c)}\right)>0$ then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{4ac-b^2}{8a^{3/2}}\ln\left(2ax+b+2\sqrt{a(ax^2+bx+c)}\right)\right]\\
&=\left(\frac{4ac-b^2}{8a^{3/2}}\right)\left[\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(2ax+b+2\sqrt{a(ax^2+bx+c)}\right)\right] && \text{by } \textit{linearity} \text{ of } \frac{\mathrm{d}}{\mathrm{d}x}\\
&=\left(\frac{4ac-b^2}{8a^{3/2}}\right)\left(\frac{1}{2ax+b+2\sqrt{a(ax^2+bx+c)}}\right)\left(2a+\frac{2\sqrt{a}(2ax+b)}{2\sqrt{ax^2+bx+c}}\right) && \text{by } \textit{chain rule}\\
&=\frac{(4ac-b^2)\left[2a\sqrt{ax^2+bx+c}+\sqrt{a}(2ax+b)\right]}{8a^{3/2}\left(2ax+b+2\sqrt{a(ax^2+bx+c)}\right)\sqrt{ax^2+bx+c}}\\
&=\frac{(4ac-b^2)\left[\sqrt{a}(2ax+b)+2a\sqrt{ax^2+bx+c}\right]}{\left(8a\sqrt{ax^2+bx+c}\right)\left[\sqrt{a}(2ax+b)+2a\sqrt{ax^2+bx+c}\right]}\\
&=\frac{4ac-b^2}{8a\sqrt{ax^2+bx+c}}
\end{aligned}$$

(3) lemma: If $\left(2ax+b+2\sqrt{a(ax^2+bx+c)}\right)<0$ then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{4ac-b^2}{8a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right|\right]\\
&=\left(\frac{4ac-b^2}{8a^{3/2}}\right)\left[\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(-2ax-b-2\sqrt{a(ax^2+bx+c)}\right)\right] && \text{by } \textit{linearity} \text{ of } \frac{\mathrm{d}}{\mathrm{d}x}\\
&=\left(\frac{4ac-b^2}{8a^{3/2}}\right)\left(\frac{1}{-2ax-b-2\sqrt{a(ax^2+bx+c)}}\right)\left(2a+\frac{2\sqrt{a}(2ax+b)}{2\sqrt{ax^2+bx+c}}\right) && \text{by } \textit{chain rule}\\
&=\frac{(4ac-b^2)\left[2a\sqrt{ax^2+bx+c}+\sqrt{a}(2ax+b)\right]}{8a^{3/2}\left(-2ax-b-2\sqrt{a(ax^2+bx+c)}\right)\sqrt{ax^2+bx+c}}\\
&=\frac{-(4ac-b^2)\left[\sqrt{a}(2ax+b)+2a\sqrt{ax^2+bx+c}\right]}{\left(8a\sqrt{ax^2+bx+c}\right)\left[\sqrt{a}(2ax+b)+2a\sqrt{ax^2+bx+c}\right]}\\
&=\frac{-(4ac-b^2)}{8a\sqrt{ax^2+bx+c}}
\end{aligned}$$

(4) Complete the proof: If $\left(2ax+b+2\sqrt{a(ax^2+bx+c)}\right)>0$ then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{2ax+b}{4a}\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8a^{3/2}}\ln\left(2ax+b+2\sqrt{a(ax^2+bx+c)}\right)\right]\\
&=\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{2ax+b}{4a}\right)\sqrt{ax^2+bx+c}\right]+\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{4ac-b^2}{8a^{3/2}}\right)\ln\left(2ax+b+2\sqrt{a(ax^2+bx+c)}\right)\right]\\
&=\frac{8a^2x^2+8abx+b^2+4ac}{8a\sqrt{ax^2+bx+c}}+\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{4ac-b^2}{8a^{3/2}}\right)\ln\left(2ax+b+2\sqrt{a(ax^2+bx+c)}\right)\right] && \text{by item (1)}\\
&=\frac{8a^2x^2+8abx+b^2+4ac}{8a\sqrt{ax^2+bx+c}}+\frac{4ac-b^2}{8a\sqrt{ax^2+bx+c}} && \text{by item (2)}\\
&=\frac{8a(ax^2+bx+c)}{8a\sqrt{ax^2+bx+c}}=\sqrt{ax^2+bx+c}
\end{aligned}$$

(5) Note that simply forcing the agruement of ln to be positive as in[18]
$\frac{2ax+b}{4a}\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right|+\varepsilon$

[18] The solution $\ln|\cdots|$ is used in Jeffrey (1995) page 160 and Jeffrey and Dai (2008) pages 172–173.

is *not* a solution to $\int \sqrt{ax^2+bx+c}\,\mathrm{d}x$ when $\left(2ax+b+2\sqrt{a(ax^2+bx+c)}\right) < 0$:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{2ax+b}{4a}\sqrt{ax^2+bx+c}+\frac{4ac-b^2}{8a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right|\right]\\
&=\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{2ax+b}{4a}\right)\sqrt{ax^2+bx+c}\right]+\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{4ac-b^2}{8a^{3/2}}\right)\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right|\right]\\
&=\frac{8a^2x^2+8abx+b^2+4ac}{8a\sqrt{ax^2+bx+c}}+\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{4ac-b^2}{8a^{3/2}}\right)\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right|\right] && \text{by item (1)}\\
&=\frac{8a^2x^2+8abx+b^2+4ac}{8a\sqrt{ax^2+bx+c}}-\frac{4ac-b^2}{8a\sqrt{ax^2+bx+c}} && \text{by item (3)}\\
&=\frac{8a(ax^2+bx)+2b^2}{8a\sqrt{ax^2+bx+c}}=\frac{8a(ax^2+bx+c)+2b^2-8ac}{8a\sqrt{ax^2+bx+c}}\\
&=\sqrt{ax^2+bx+c}+\frac{2b^2-8ac}{8a\sqrt{ax^2+bx+c}}\neq\sqrt{ax^2+bx+c}\quad\text{for } b^2\neq 4ac
\end{aligned}$$

(6) Note further that constraining $a>0$ is also not a solution[19] because it does not guarantee that the argument u of $\ln(u)$ will be positive. Take for example $a=1$, $b=-3$, $c=3$ and $x=1$. Then
$$2ax+b+2\sqrt{a(ax^2+bx+c)}=2\cdot1\cdot1-3+2\sqrt{1(1\cdot1-3\cdot1+3)}=-1<0\,.$$

✏

Theorem 3.3 *Let* $\mathsf{R}(p,q)$, r_p, r_q, *and* ϕ *be as defined in Definition 3.1 (page 7). Let* $\rho \triangleq r_q - r_p$. *If* $r_p \neq 0$, $r_q \neq 0$ *and* $\phi \neq 0$ *then*

$$\mathsf{R}(p,q) = \frac{r_q\sqrt{(r_q\phi)^2+\rho^2}-r_p\sqrt{(r_p\phi)^2+\rho^2}}{2\rho}+\frac{|\rho|}{2\phi}\left[\ln\left(r_q\rho\phi+|\rho|\sqrt{(r_q\phi)^2+\rho^2}\right)-\ln\left(r_p\rho\phi+|\rho|\sqrt{(r_p\phi)^2+\rho^2}\right)\right]$$

✎PROOF:

(1) Let $\gamma \triangleq r_p\phi$.

(2) lemmas:

$$\begin{aligned}
\rho\phi+\gamma &= (r_q-r_p)\phi+r_p\phi=r_q\phi\\
\rho^2\phi^2+2\rho\gamma\phi+(\gamma^2+\rho^2) &= (r_q-r_p)^2\phi^2+2(r_q-r_p)(r_p\phi)\phi+(r_p\phi)^2+\rho^2\\
&=(r_p^2+r_q^2-2r_pr_q)\phi^2+2(r_q-r_p)(r_p\phi)\phi+(r_p\phi)^2+\rho^2\\
&=(r_p^2+r_q^2)\phi^2-2(r_p\phi)^2+(r_p\phi)^2+\rho^2\\
&=(r_q\phi)^2+\rho^2
\end{aligned}$$

(3) lemma: $2\rho^2\theta+2\rho\gamma+2\sqrt{\rho^2(\rho^2\theta^2+2\rho\gamma\theta+(\gamma^2+\rho^2))}>0$. Proof:

$$\begin{aligned}
&2\rho^2\theta+2\rho\gamma+2\sqrt{\rho^2(\rho^2\theta^2+2\rho\gamma\theta+(\gamma^2+\rho^2))}\\
&\quad>2\rho^2\theta+2\rho\gamma+2\sqrt{\rho^2\gamma^2} && \text{because } \sqrt{x} \text{ is strictly monotonically increasing}\\
&\quad=2\rho^2\theta+2\rho\gamma+2|\rho|\gamma && \text{because } \gamma>0\\
&\quad=2\rho^2\theta+2\gamma(\rho+|\rho|)\\
&\quad\geq 0
\end{aligned}$$

[19] The $a>0$ constraint is used in 📚 Gradshteyn and Ryzhik (2007) page 94

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

(4)

$$\begin{aligned}
\mathsf{R}(p,q) &\triangleq \int_{\theta=0}^{\theta=\phi} \sqrt{r^2(\theta) + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta && \text{by def. of } \mathsf{R}(p,q)\\
&= \int_0^{\phi} \sqrt{\left[\left(\frac{r_q - r_p}{\phi}\right)\theta + r_p\right]^2 + \left[\frac{r_q - r_p}{\phi}\right]^2}\,\mathrm{d}\theta && \text{by def. of } \mathsf{r}(\theta)\\
&= \int_0^{\phi} \sqrt{\left[\frac{(r_q - r_p)\theta + r_p\phi}{\phi}\right]^2 + \left[\frac{r_q - r_p}{\phi}\right]^2}\,\mathrm{d}\theta\\
&= \frac{1}{\phi}\int_0^{\phi} \sqrt{(r_q - r_p)^2\theta^2 + 2(r_q - r_p)(r_p\phi)\theta + (r_p\phi)^2 + (r_q - r_p)^2}\,\mathrm{d}\theta\\
&= \frac{1}{\phi}\int_0^{\phi} \sqrt{\underbrace{\rho^2}_{a}\theta^2 + \underbrace{2\rho\gamma}_{b}\theta + \underbrace{(\gamma^2 + \rho^2)}_{c}}\,\mathrm{d}\theta\\
&= \frac{1}{\phi}\left[\frac{2\rho^2\theta + 2\rho\gamma}{4\rho^2}\sqrt{\rho^2\theta^2 + 2\rho\gamma\theta + (\gamma^2 + \rho^2)}\right. && \text{[by item (3) and}\\
&\quad \left. + \frac{4\rho^2(\gamma^2 + \rho^2) - (2\rho\gamma)^2}{8|\rho|^3}\ln\left(2\rho^2\theta + 2\rho\gamma + 2\sqrt{\rho^2(\rho^2\theta^2 + 2\rho\gamma\theta + (\gamma^2 + \rho^2))}\right)\right]_{\theta=0}^{\theta=\phi} && \text{by Lemma 3.2]}\\
&= \frac{1}{\phi}\left[\frac{\rho\theta + \gamma}{2\rho}\sqrt{\rho^2\theta^2 + 2\rho\gamma\theta + (\gamma^2 + \rho^2)} + \frac{|\rho|}{2}\ln\left(2\rho^2\theta + 2\rho\gamma + 2|\rho|\sqrt{\rho^2\theta^2 + 2\rho\gamma\theta + (\gamma^2 + \rho^2)}\right)\right]_{\theta=0}^{\theta=\phi}\\
&= \left[\frac{(\rho\phi + \gamma)\sqrt{\rho^2\phi^2 + 2\rho\gamma\phi + (\gamma^2 + \rho^2)}}{2\rho\phi} + \frac{|\rho|}{2\phi}\ln\left(2\rho^2\phi + 2\rho\gamma + 2|\rho|\sqrt{\rho^2\phi^2 + 2\rho\gamma\phi + (\gamma^2 + \rho^2)}\right)\right]\\
&\quad - \left[\frac{\gamma\sqrt{\gamma^2 + \rho^2}}{2\rho\phi} + \frac{|\rho|}{2\phi}\ln\left(2\rho\gamma + 2|\rho|\sqrt{\gamma^2 + \rho^2}\right)\right]\\
&= \frac{r_q\phi\sqrt{(r_q\phi)^2 + \rho^2} - r_p\phi\sqrt{(r_p\phi)^2 + \rho^2}}{2\rho\phi}\\
&\quad + \frac{|\rho|}{2\phi}\left[\ln(2) + \ln\left(\rho^2\phi + \rho\gamma + |\rho|\sqrt{(r_q\phi)^2 + \rho^2}\right) - \ln(2) - \ln\left(\rho\gamma + |\rho|\sqrt{\gamma^2 + \rho^2}\right)\right] && \text{by item (2)}\\
&= \frac{r_q\sqrt{(r_q\phi)^2 + \rho^2} - r_p\sqrt{(r_p\phi)^2 + \rho^2}}{2\rho} + \frac{|\rho|}{2\phi}\left[\ln\left(r_q\rho\phi + |\rho|\sqrt{(r_q\phi)^2 + \rho^2}\right) - \ln\left(r_p\rho\phi + |\rho|\sqrt{(r_p\phi)^2 + \rho^2}\right)\right]
\end{aligned}$$

✏

3.3 Arc function R(p,q) properties

If we really want the *Langrange arc distance* $\mathsf{d}(p,q)$ to be an *extension* of the *spherical metric*, then $\mathsf{R}(p,q)$ must equal $r_p\phi$ (the arc length between p and q on a circle centered at the origin) when $r_p = r_q$. This is in fact the case, as demonstrated next.

Proposition 3.4 ($\mathsf{R}(p,q)$ on spherical surface) *Let* $\mathsf{R}(p,q)$, r_p, r_q, *and* ϕ *be defined as in Definition 3.1.*

$$\left\{\begin{array}{rcll} r_p = r_q & \neq & 0 & \text{and}\\ \phi & \neq & 0 & \end{array}\right\} \implies \left\{\mathsf{R}(p,q) = r_p\phi\right\}$$

✎PROOF:

(1) lemma:

$$\begin{aligned}
\lim_{\rho\to 0}\frac{r_q\sqrt{(r_q\phi)^2 + \rho^2} - r_p\sqrt{(r_p\phi)^2 + \rho^2}}{2\rho} &= \lim_{\rho\to 0}\frac{r_q\sqrt{(r_q\phi)^2 + 0} - r_p\sqrt{(r_p\phi)^2 + 0}}{2\rho} &&= \lim_{\rho\to 0}\left(\frac{\phi}{2}\right)\frac{r_q^2 - r_p^2}{r_q - r_p}\\
&= \lim_{\rho\to 0}\left(\frac{\phi}{2}\right)\frac{(r_q - r_p)(r_q + r_p)}{r_q - r_p} &&= \lim_{\rho\to 0}\left(\frac{\phi}{2}\right)(r_q + r_p)\\
&= r_p\phi
\end{aligned}$$

(2) lemma:

$$\begin{aligned}
&\lim_{\rho\to0^+}\frac{\rho}{2\phi}\left[\ln\left(r_q\rho\phi+|\rho|\sqrt{(r_q\phi)^2+\rho^2}\right)-\ln\left(r_p\rho\phi+|\rho|\sqrt{(r_p\phi)^2+\rho^2}\right)\right]\\
&=\lim_{\rho\to0^+}\frac{\rho}{2\phi}\left[\ln\left(r_q\rho\phi+\rho\sqrt{(r_q\phi)^2}\right)-\ln\left(r_p\rho\phi+\rho\sqrt{(r_p\phi)^2}\right)\right]\\
&=\lim_{\rho\to0^+}\frac{\rho}{2\phi}\left[\ln(\rho)+\ln\left(r_q\phi+\sqrt{(r_q\phi)^2}\right)-\ln(\rho)-\ln\left(r_p\phi+\sqrt{(r_p\phi)^2}\right)\right]\\
&=\lim_{\rho\to0^+}\frac{\rho}{2\phi}\left[+\ln\left(r_q\phi+\sqrt{(r_q\phi)^2}\right)-\ln\left(r_p\phi+\sqrt{(r_p\phi)^2}\right)\right]\\
&=0
\end{aligned}$$

(3) lemma:

$$\begin{aligned}
&\lim_{\rho\to0^-}\frac{\rho}{2\phi}\left[\ln\left(r_q\rho\phi+|\rho|\sqrt{(r_q\phi)^2+\rho^2}\right)-\ln\left(r_p\rho\phi+|\rho|\sqrt{(r_p\phi)^2+\rho^2}\right)\right]\\
&=\lim_{\rho\to0^-}\frac{\rho}{2\phi}\left[\ln\left(-r_q|\rho|\phi+|\rho|\sqrt{(r_q\phi)^2}\right)-\ln\left(-r_p|\rho|\phi+|\rho|\sqrt{(r_p\phi)^2}\right)\right]\\
&=\lim_{\rho\to0^-}\frac{\rho}{2\phi}\left[\ln(\rho)+\ln\left(-r_q\phi+\sqrt{(r_q\phi)^2}\right)-\ln(\rho)-\ln\left(-r_p\phi+\sqrt{(r_p\phi)^2}\right)\right]\\
&=\lim_{\rho\to0^-}\frac{\rho}{2\phi}\left[\ln\left(-r_q\phi+\sqrt{(r_q\phi)^2}\right)-\ln\left(-r_p\phi+\sqrt{(r_p\phi)^2}\right)\right]\\
&=0
\end{aligned}$$

(4) lemma: By item (2), item (3), and by *continuity* ...

$$\lim_{\rho\to0}\frac{\rho}{2\phi}\left[\ln\left(r_q\rho\phi+|\rho|\sqrt{(r_q\phi)^2+\rho^2}\right)-\ln\left(r_p\rho\phi+|\rho|\sqrt{(r_p\phi)^2+\rho^2}\right)\right]=0$$

(5) Completing the proof ...

$$\begin{aligned}
\lim_{\rho\to0}\mathsf{R}(p,q)&=\lim_{\rho\to0}\left[\frac{r_q\sqrt{(r_q\phi)^2+\rho^2}-r_p\sqrt{(r_p\phi)^2+\rho^2}}{2\rho}\right.\\
&\qquad\left.+\frac{\rho}{2\phi}\ln\left(r_q\rho\phi+|\rho|\sqrt{(r_q\phi)^2+\rho^2}\right)-\ln\left(r_p\rho\phi+|\rho|\sqrt{(r_p\phi)^2+\rho^2}\right)\right] &&\text{by Theorem 3.3}\\
&=0+\lim_{\rho\to0}\frac{\rho}{2\phi}\left[\ln\left(r_q\rho\phi+|\rho|\sqrt{(r_q\phi)^2+\rho^2}\right)-\ln\left(r_p\rho\phi+|\rho|\sqrt{(r_p\phi)^2+\rho^2}\right)\right] &&\text{by item (1)}\\
&=0+0 &&\text{by item (4)}\\
&=0
\end{aligned}$$

✏

Later in Theorem 3.12 (page 15), we want to prove that the *Langrange arc distance* (Definition 3.1 page 7) $\mathsf{d}(p,q)$ is indeed, as its name suggests, a *distance function* (Definition A.1 page 21). Proposition 3.5 (*symmetry*) and Proposition 3.7 (*positivity*) will help. Meanwhile, Proposition 3.7 will itself receive help from Proposition 3.6 (*monotonicity*).

Proposition 3.5 (*symmetry* of R) *Let* $\mathsf{R}(p,q)$ *be defined as in Definition 3.1 (page 7).*

$$\mathsf{R}(p,q) = \mathsf{R}(q,p) \quad \forall p,q\in\mathbb{R}^N \quad \text{(symmetric)}$$

✎PROOF:

(1) dummy variable: Let $\mu \triangleq \phi-\theta$ which implies $\theta=\phi-\mu$ and $\mathsf{d}\theta = -\,\mathsf{d}\mu$.

(2) lemma:

$$\begin{aligned}
\mathsf{r}(\mu;q,p)&\triangleq\mathsf{r}(\phi-\theta;q,p) &&\text{by item (1)}\\
&=\left(\frac{r_p-r_q}{\phi}\right)(\phi-\theta)+r_q=\left(\frac{r_q-r_p}{\phi}\right)\theta+\left(r_p-r_q\right)+r_q=\left(\frac{r_q-r_p}{\phi}\right)\theta+r_p\\
&=\mathsf{r}(\theta;p,q)
\end{aligned}$$

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

(3) Completing the proof…

$$\begin{aligned}
\mathsf{R}(p,q) &= \int_{\theta=0}^{\theta=\phi} \sqrt{\mathsf{r}^2(\theta;p,q) + \left[\frac{\mathsf{d}\mathsf{r}(\theta;p,q)}{\mathsf{d}\theta}\right]^2}\,\mathsf{d}\theta \\
&= \int_{\phi-\mu=0}^{\phi-\mu=\phi} \sqrt{\mathsf{r}^2(\mu;q,p) + \left[\frac{\mathsf{d}\mathsf{r}(\mu;q,p)}{-\,\mathsf{d}\mu}\right]^2}\,(-\,\mathsf{d}\mu) && \text{by item (1) and item (2)} \\
&= -\int_{\mu=\phi}^{\mu=0} \sqrt{\mathsf{r}^2(\mu;q,p) + \left[\frac{\mathsf{d}\mathsf{r}(\mu;q,p)}{\mathsf{d}\mu}\right]^2}\,\mathsf{d}\mu \\
&= \int_{\mu=0}^{\mu=\phi} \sqrt{\mathsf{r}^2(\mu;q,p) + \left[\frac{\mathsf{d}\mathsf{r}(\mu;q,p)}{\mathsf{d}\mu}\right]^2}\,\mathsf{d}\mu && \text{by the \textit{Second Fundamental Theorem of Calculus}}^{20} \\
&= \mathsf{R}(q,p)
\end{aligned}$$

✏

Proposition 3.6 (*monotonicity* of R) *Let* $\mathsf{R}(p,q)$ *and* ϕ *be defined as in Definition 3.1 (page 7). Let* ϕ_1 *be the polar angle between the point pair* (p_1,q_1) *in* $\mathbb{R}^N$ *and Let* ϕ_2 *the polar angle between the point pair* (p_2,q_2) *in* $\mathbb{R}^N$*.*

$$\{\phi_1 < \phi_2\} \implies \{\mathsf{R}(p_1,q_1) < \mathsf{R}(p_2,q_2)\} \quad \forall \phi_1,\phi_2 \in (0:\pi] \quad \text{(STRICTLY MONOTONICALLY INCREASING } in\ \phi)$$

✎PROOF:

$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}\phi}\mathsf{R}(p,q) &\triangleq \frac{\mathsf{d}}{\mathsf{d}\phi}\int_0^{\phi} \sqrt{\mathsf{r}^2(\theta) + \left[\frac{\mathsf{d}\mathsf{r}(\theta)}{\mathsf{d}\theta}\right]^2}\,\mathsf{d}\theta && \text{by Definition 3.1 (page 7)} \\
&= \frac{\mathsf{d}}{\mathsf{d}\phi}\int_0^{\phi} \sqrt{\left[\left(\frac{r_q-r_p}{\phi}\right)\theta + r_p\right]^2 + \left[\frac{r_q-r_p}{\phi}\right]^2}\,\mathsf{d}\theta \\
&= \sqrt{\left[\left(\frac{r_q-r_p}{\phi}\right)\phi + r_p\right]^2 + \left[\frac{r_q-r_p}{\phi}\right]^2} && \text{by the \textit{First Fundamental Theorem of Calculus}}^{21} \\
&= \sqrt{r_q^2 + \left[\frac{r_q-r_p}{\phi}\right]^2} \\
&> 0 \\
&\implies \mathsf{R}(p,q) \text{ is \textit{strictly monotonically increasing} in } \phi
\end{aligned}$$

✏

Proposition 3.7 (*positivity* of R) *Let* $\mathsf{R}(p,q)$ *and* ϕ *be defined as in Definition 3.1 (page 7).*

$$\left\{\begin{array}{ccc} \phi & \in & (0:\pi] \\ (\phi & \neq & 0) \end{array}\right\} \implies \left\{ \mathsf{R}(p,q) > 0 \quad \forall p,q \in \mathbb{R}^N \quad \textit{(positive)} \right\}$$

✎PROOF:

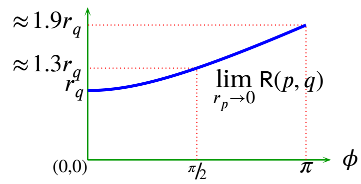
$$\begin{aligned}
\frac{\mathsf{d}}{\mathsf{d}\phi}\mathsf{R}(p,q) &\triangleq \int_0^{\phi} \sqrt{\mathsf{r}^2(\theta) + \left[\frac{\mathsf{d}\mathsf{r}(\theta)}{\mathsf{d}\theta}\right]^2}\,\mathsf{d}\theta && \text{by Definition 3.1 (page 7)} \\
&= \int \sqrt{\mathsf{r}^2(\theta) + \left[\frac{\mathsf{d}\mathsf{r}(\theta)}{\mathsf{d}\theta}\right]^2}\,\mathsf{d}\theta\Bigg|_{\phi} - \int \sqrt{\mathsf{r}^2(\theta) + \left[\frac{\mathsf{d}\mathsf{r}(\theta)}{\mathsf{d}\theta}\right]^2}\,\mathsf{d}\theta\Bigg|_{0} && \text{by the Second Fundamental Theorem of Calculus} \\
&> 0 && \text{by Proposition 3.6 (page 12)}
\end{aligned}$$

[20] Hijab (2016) page 170 ⟨Theorem 4.4.3 Second Fundamental Theorem of Calculus⟩, Amann and Escher (2008) page 31 ⟨Theorem 4.13 The second fundamental theorem of calculus⟩

[21] Schechter (1996) page 674 ⟨25.15⟩, Haaser and Sullivan (1991) page 218

✏

For the sake of *continuity* at the origin of $\mathbb{R}^N$, one might hope that it doesn't matter which "direction" the points p or q approach the origin when computing the limit of $\mathsf{R}(p,q)$. This however is *not* the case, as demonstrated next and illustrated to the right and in Example 3.9 (page 13). In fact, the limits very much depend on ϕ…resulting in a *discontinuity* at the origin, as demonstrated in Theorem 3.11 (page 15).



Proposition 3.8 (limit cases of R) *Let $\mathsf{R}(p,q)$, r_p, r_q, and ϕ be defined as in Definition 3.1 (page 7).*

$$\lim_{r_p\to 0} \mathsf{R}(p,q) = \frac{r_q}{2}\left[\sqrt{\phi^2+1} + \frac{\ln\left(\phi+\sqrt{\phi^2+1}\right)}{\phi}\right] \quad \forall p,q\in\mathbb{R}^N\backslash(0,0,\cdots,0),\ \phi\neq 0 \quad \text{(p approaching origin)}$$

$$\lim_{r_q\to 0} \mathsf{R}(p,q) = \frac{r_p}{2}\left[\sqrt{\phi^2+1} + \frac{\ln\left(\phi+\sqrt{\phi^2+1}\right)}{\phi}\right] \quad \forall p,q\in\mathbb{R}^N\backslash(0,0,\cdots,0),\ \phi\neq 0 \quad \text{(q approaching origin)}$$

✎PROOF:

$$\lim_{r_p\to 0} \mathsf{R}(p,q) = \lim_{r_p\to 0} \frac{r_q\sqrt{(r_q\phi)^2+\rho^2} - r_p\sqrt{(r_p\phi)^2+\rho^2}}{2\rho} + \frac{|\rho|}{2\phi}\left[\ln\left(r_q\rho\phi+|\rho|\sqrt{(r_q\phi)^2+\rho^2}\right) - \ln\left(r_p\rho\phi+|\rho|\sqrt{(r_p\phi)^2+\rho^2}\right)\right]$$

by Theorem 3.3 (page 9)

$$= \frac{r_q\sqrt{(r_q\phi)^2+r_q^2} - 0}{2r_q} + \frac{|r_q|}{2\phi}\left[\ln\left(r_q^2\phi+|r_q|\sqrt{(r_q\phi)^2+r_q^2}\right) - \ln\left(0+|r_q|\sqrt{0+r_q^2}\right)\right] \quad \text{by } \lim_{r_p\to 0} \text{ operation}$$

$$= \frac{r_q\sqrt{\phi^2+1}}{2} + \frac{r_q}{2\phi}\left[\ln\left(r_q^2\phi+r_q^2\sqrt{\phi^2+1}\right) - \ln\left(r_q^2\right)\right]$$

$$= \frac{r_q}{2}\left[\sqrt{\phi^2+1} + \frac{\ln(r_q^2)+\ln\left(\phi+\sqrt{\phi^2+1}\right)-\ln\left(r_q^2\right)}{\phi}\right] = \frac{r_q}{2}\left[\sqrt{\phi^2+1} + \frac{\ln\left(\phi+\sqrt{\phi^2+1}\right)}{\phi}\right]$$

$$\lim_{r_q\to 0} \mathsf{R}(p,q) = \lim_{r_q\to 0} \mathsf{R}(q,p) \quad \text{by Proposition 3.5}$$

$$= \frac{r_p}{2}\left[\sqrt{\phi^2+1} + \frac{\ln\left(\phi+\sqrt{\phi^2+1}\right)}{\phi}\right] \quad \text{by previous result}$$

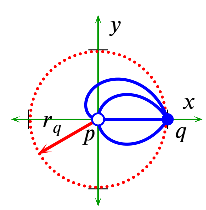
✏

Example 3.9 Let $\mathsf{R}(p,q)$, ϕ, and r_q be defined as in Definition 3.1.

$$\text{If } \phi=0 \text{ then } \lim_{p\to 0} \mathsf{R}\left[p,(r_q,0)\right] = r_q$$

$$\text{If } \phi=\pi/2 \text{ then } \lim_{p\to 0} \mathsf{R}\left[p,(r_q,0)\right] = r_q\times(1.323652\cdots) \approx 1.3r_q$$

$$\text{If } \phi=\pi \text{ then } \lim_{p\to 0} \mathsf{R}\left[p,(r_q,0)\right] = r_q\times(1.944847\cdots) \approx 1.9r_q$$



✎PROOF:

$$\mathsf{R}(p,q)|_{\phi=\pi/2} = \frac{r_q}{2}\left[\sqrt{\left(\frac{\pi}{2}\right)^2+1} + \frac{\ln\left(\frac{\pi}{2}+\sqrt{\left(\frac{\pi}{2}\right)^2+1}\right)}{\frac{\pi}{2}}\right] = r_q\times(1.323652\cdots)$$

$$\mathsf{R}(p,q)|_{\phi=\pi} = \frac{r_q}{2}\left[\sqrt{\pi^2+1} + \frac{\ln\left(\pi+\sqrt{\pi^2+1}\right)}{\pi}\right] = r_q\times(1.944847\cdots)$$

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

✏

3.4 Distance function d(p,q) properties

The *Langrange arc distance* $\mathsf{d}(p,q)$ is defined in two parts: one part being the Euclidean distance $^1\!/\!_\pi |r_q - r_p|$ and the second part the length of the arc $^1\!/\!_\pi \mathsf{R}(p,q)$. There is risk in creating a multipart definition...with the possible consequences being *discontinuity* at the boundary of the parts. Proposition 3.10 (next) demonstrates that when $r_p \neq 0$ and $r_q \neq 0$, there is *continuity* as $\phi \to 0$. However, Theorem 3.12 (page 15) demonstrates that in general for values of $\phi > 0$, $\mathsf{d}(p,q)$ is *discontinuous* at the *origin*.

Proposition 3.10 *Let* $\mathsf{R}(p,q)$, r_p, r_q, ϕ, *and* $(0,0,\cdots,0)$ *be defined as in Definition 3.1 (page 7).*

$$\text{(A).}\quad \lim_{\phi\to 0}\mathsf{R}(p,q) = |r_q - r_p| = \pi\mathsf{d}(p,q) \quad \textit{when } \phi = 0$$

$$\text{(B).}\quad \lim_{\phi\to 0}\lim_{r_p\to 0}\mathsf{R}(p,q) = \lim_{r_p\to 0}\lim_{\phi\to 0}\mathsf{R}(p,q) = r_q = \pi\mathsf{d}((0,0,\cdots,0),q)$$

$$\text{(C).}\quad \lim_{\phi\to 0}\lim_{r_q\to 0}\mathsf{R}(p,q) = \lim_{r_q\to 0}\lim_{\phi\to 0}\mathsf{R}(p,q) = r_p = \pi\mathsf{d}(p,(0,0,\cdots,0))$$

✎PROOF:

$$\begin{aligned}
\lim_{\phi\to 0}\mathsf{R}(p,q) &\triangleq \lim_{\phi\to 0}\int_0^\phi \sqrt{r^2(\theta) + \left(\frac{\mathsf{d}r}{\mathsf{d}\theta}\right)^2}\,\mathsf{d}\theta && \text{by definition of } \mathsf{R}(p,q) \text{ (Definition 3.1 page 7)}\\
&= \lim_{\phi\to 0}\int_0^\phi \sqrt{\left[\left(\frac{r_q - r_p}{\phi}\right)\theta + r_p\right]^2 + \left[\frac{r_q - r_p}{\phi}\right]^2}\,\mathsf{d}\theta && \text{by definition of } \mathsf{r}(\theta) \text{ (Definition 3.1 page 7)}\\
&= \lim_{\phi\to 0}\left[\frac{1}{\phi}\int_0^\phi \sqrt{[(r_q - r_p)\theta + r_p\phi]^2 + [r_q - r_p]^2}\,\mathsf{d}\theta\right]\\
&= \frac{\lim_{\phi\to 0}\frac{\mathsf{d}}{\mathsf{d}\phi}\int_0^\phi \sqrt{[(r_q - r_p)\theta + r_p\phi]^2 + [r_q - r_p]^2}\,\mathsf{d}\theta}{\lim_{\phi\to 0}\frac{\mathsf{d}}{\mathsf{d}\phi}\phi} && \text{by \textit{L'Hôpital's rule}}\\
&= \frac{\lim_{\phi\to 0}\sqrt{[(r_q - r_p)\phi + r_p\phi]^2 + [r_q - r_p]^2}}{1} && \text{by \textit{First Fundamental Theorem of Calculus}}\\
&= \sqrt{[r_q - r_p]^2}\\
&= |r_q - r_p|\\
&= \pi\,\mathsf{d}(p,q)|_{\phi=0} && \text{by definition of \textit{Lagrange arc distance} (Definition 3.1)}
\end{aligned}$$

$$\begin{aligned}
\lim_{\phi\to 0}\lim_{r_p\to 0}\mathsf{R}(p,q) &= \lim_{\phi\to 0}\frac{r_q}{2}\left[\sqrt{\phi^2+1} + \frac{\ln\left(\phi + \sqrt{\phi^2+1}\right)}{\phi}\right] && \text{by Proposition 3.8 (page 13)}\\
&= \frac{r_q}{2}\left[\sqrt{0^2+1} + \frac{\lim_{\phi\to 0}\frac{\mathsf{d}}{\mathsf{d}\phi}\ln\left(\phi+\sqrt{\phi^2+1}\right)}{\lim_{\phi\to 0}\frac{\mathsf{d}}{\mathsf{d}\phi}\phi}\right] && \text{by \textit{L'Hôpital's rule}}\\
&= \frac{r_q}{2}\left[1 + \lim_{\phi\to 0}\frac{1 + \frac{2\phi}{2\sqrt{\phi^2+1}}}{\phi + \sqrt{\phi^2+1}}\right] = \frac{r_q}{2}[1 + 1/1] = r_q = \pi\mathsf{d}((0,0,\cdots,0),q)
\end{aligned}$$

$$\begin{aligned}
\lim_{\phi\to 0}\lim_{r_q\to 0}\mathsf{R}(p,q) &= \lim_{\phi\to 0}\lim_{r_q\to 0}\mathsf{R}(q,p) && \text{by Proposition 3.5 (page 11)}\\
&= r_p = \pi\mathsf{d}(p,(0,0,\cdots,0)) && \text{by previous result}\\
\lim_{r_p\to 0}\lim_{\phi\to 0}\mathsf{R}(p,q) &= \lim_{r_p\to 0}|r_q - r_p| && \text{by (A)}
\end{aligned}$$

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

$$= r_q = \pi \mathsf{d}((0,0,\cdots,0), q)$$

$$\lim_{r_q\to 0}\lim_{\phi\to 0} \mathsf{R}(p,q) = \lim_{r_q\to 0} |r_q - r_p| \qquad \text{by (A)}$$

$$= r_p = \pi \mathsf{d}(p, (0,0,\cdots,0))$$

✏

Theorem 3.11 *Let the Lagrange arc distance* $\mathsf{d}(p,q)$ *and origin be defined as in Definition 3.1. The function* $\mathsf{d}(p,q)$ *is discontinuous at the origin of* $\mathbb{R}^N$*, but is continuous everywhere else in* $\mathbb{R}^N$*.*

✎PROOF:

(1) Proof for when p and q are *not* at the origin and $\phi \neq 0$:
 (a) In this case, $\mathsf{d}(p,q) = {}^1\!/\!_\pi \mathsf{R}(p,q)$.
 (b) $\mathsf{R}(p,q)$ is continuous everywhere in its domain because its solution, as given by Theorem 3.3 (page 9), consists entirely of continuous functions such as $\ln(x)$, $|x|$, etc.
 (c) Therefore, in this case, $\mathsf{d}(p,q)$ is also *continuous*.

(2) Proof for when p and q are *not* at the origin and $\phi = 0$:
This follows from (A) of Proposition 3.10 (page 14).

(3) Proof for *discontinuity* at origin: This follows from Proposition 3.8 (page 13), where it is demonstrated that the limit of $\mathsf{R}(p,q)$ is very much dependent on the "direction" from which p or q approaches the origin. For an illustration of this concept, see Example 3.9 (page 13).

✏

Theorem 3.12 *Let* $\mathsf{d}(p,q)$ *be Lagrange arc distance* (Definition 3.1 page 7)*.*
The function $\mathsf{d}(p,q)$ *is a distance function* (Definition A.1 page 21)*. In particular,*

(1).	$\mathsf{d}(p,q)$	$\geq$	0	$\forall p,q\in\mathbb{R}^N$	*(non-negative)*	*and*
(2).	$\mathsf{d}(p,q)$	$=$	$0 \iff p = q$	$\forall p,q\in\mathbb{R}^N$	*(nondegenerate)*	*and*
(3).	$\mathsf{d}(p,q)$	$=$	$\mathsf{d}(q,p)$	$\forall p,q\in\mathbb{R}^N$	*(symmetric)*	

✎PROOF: The *Langrange arc distance* (Definition 3.1 page 7) is simply the *Euclidean metric* (Definition B.3 page 22) if p or q is at the origin, or if $\phi = 0$. In this case, (1)–(3) are satisfied automatically because all *metric*s have these properties (Definition B.1 page 22). What is left to prove is that $\mathsf{R}(p,q)$ has these properties when p and q are not at the origin and $\phi \neq 0$.

(1) Proof that $\mathsf{R}(p,q) \geq 0$: If $\phi = 0$, then the Euclidean metric is used.
For any $\phi > 0$, $\mathsf{R}(p,q) > 0$, as demonstrated by Proposition 3.7 (page 12).

(2) Proof that $p = q \implies \mathsf{R}(p,q) = 0$: If $p = q$, then $\phi = 0$, and the Euclidean metric is used, not $\mathsf{R}(p,q)$.

(3) Proof that $\mathsf{R}(p,q) = 0 \implies p = q$: If $\mathsf{d}(p,q) = 0$ and $\phi = 0$, then the Euclidean metric is used. If $\mathsf{d}(p,q) = 0$ and $\phi > 0$, then $\mathsf{R}(p,q)$ never equals 0 anyways, as demonstrated by Proposition 3.7 (page 12).

(4) Proof that $\mathsf{R}(p,q) = \mathsf{R}(q,p)$: This is demonstrated by Proposition 3.5 (page 11).

✏

The *Lagrange arc distance* is *not* a *metric* because in general the *triangle inequality* property does not hold (next theorem). Furthermore, the *Lagrange arc distance* does not induce a *norm* because it is *not translation invariant* (the *translation invariant* property is a necessary condition for a *metric* to induce a *norm*, Theorem D.14 page 30), and balls in a *Lagrange arc distance space* are in general *not convex* (balls are always *convex* in a *normed linear space*, Theorem D.12 page 28). For more details about *distance space*s, see APPENDIX A (page 21).

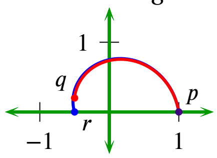
Theorem 3.13 *In the Lagrange arc distance space* (X,d) *over a field* $\mathbb{F}$

(1).	$\mathsf{d}(p,r)$	$\nleq$	$\mathsf{d}(p,q)+\mathsf{d}(q,r)$	$\forall p,q,r\in X$	*(triangle inequality fails)*
(2).	$\mathsf{d}(p+r,q+r)$	$\neq$	$\mathsf{d}(p,q)$	$\forall p,q,r\in X$	*(not translation invariant)*
(3).	$\mathsf{d}(\alpha p,\alpha q)$	$=$	$|\alpha|\,\mathsf{d}(p,q)$	$\forall p,q,r\in X,\ \alpha\in\mathbb{R}$	*(homogeneous)*

(4). d *does not induce a norm*
(5). *balls in* (X,d) *are in general not convex*

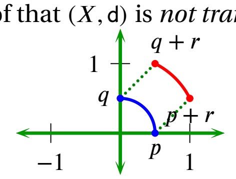
✎PROOF:

(1) Proof that the *triangle inequality* property fails to hold in (X, d): Consider the following case[22]…



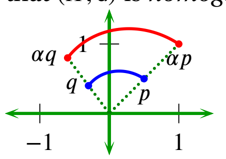
$$\begin{aligned} \mathsf{d}(p, r) &\triangleq \mathsf{d}((1,0),(-0.5,0)) &&= 0.767324\cdots \\ &\not\leq 0.756406\cdots &&= 0.692330\cdots + 0.064076\cdots \\ &= \mathsf{d}((1,0),(-0.5,0.2)) &&+ \mathsf{d}((-0.5,0.2),(-0.5,0)) \\ &\triangleq \mathsf{d}(p,q) + \mathsf{d}(q,r) \\ &\implies \textit{triangle inequality fails in } (X, \mathsf{d}) \end{aligned}$$

(2) Proof that (X, d) is *not translation invariant*:[23] Let $r \triangleq (1/2, 1/2)$. Then…



$$\begin{aligned} \mathsf{d}(p + r, q + r) &\triangleq \mathsf{d}\Big(\big(1, \tfrac{1}{2}\big), \big(\tfrac{1}{2}, 1\big)\Big) = 0.229009\cdots \neq \tfrac{1}{2} \\ &= \mathsf{d}\Big(\big(\tfrac{1}{2}, 0\big), \big(0, \tfrac{1}{2}\big)\Big) \\ &\triangleq \mathsf{d}(p, q) \\ &\implies (X, \mathsf{d}) \textit{ is not translation invariant} \end{aligned}$$

(3) Proof that (X, d) is *homogeneous*:



Let $r_{\alpha p}$ be the magnitude of $\alpha p \triangleq (\!(\alpha x_1, \alpha x_2, \cdots, \alpha x_N)\!)$.
Let $r_{\alpha q}$ be the magnitude of $\alpha q \triangleq (\!(\alpha y_1, \alpha y_2, \cdots, \alpha y_N)\!)$.
Let ϕ_α be the *polar angle* between αp and αq.

(a) If $r_p = 0$ or $r_q = 0$ or $\phi = 0$ then $\mathsf{d}(p, q)$ is the *Euclidean metric*, which is *homogeneous*.

(b) lemmas:

$$\begin{aligned} r_{\alpha p} &\triangleq \left(\sum_1^N [\alpha x_n]^2\right)^{\frac{1}{2}} &&= |\alpha| \sum_1^N x_n^2 &&\triangleq |\alpha| r_p \\ r_{\alpha q} &\triangleq \left(\sum_1^N [\alpha y_n]^2\right)^{\frac{1}{2}} &&= |\alpha| \sum_1^N y_n^2 &&\triangleq |\alpha| r_q \\ \phi_\alpha &\triangleq \arccos\left(\frac{1}{r_{\alpha p} r_{\alpha q}} \sum_{n=1}^N [\alpha x_n][\alpha y_n]\right) &&= \arccos\left(\frac{1}{r_p r_q} \sum_{n=1}^N x_n y_n\right) &&\triangleq \phi \\ \mathsf{r}(\theta; \alpha p, \alpha q) &\triangleq \left(\frac{r_{\alpha q} - r_{\alpha p}}{\phi_\alpha}\right)\theta + r_{\alpha p} &&= \left(\frac{\alpha r_q - \alpha r_p}{\phi}\right)\theta + \alpha r_p &&= \alpha \mathsf{r}(\theta; p, q) \end{aligned}$$

(c) If $\mathsf{d}(p, q)$ is *not* the *Euclidean metric* then …

$$\begin{aligned} \pi \mathsf{d}(\alpha p, \alpha q) &\triangleq \mathsf{R}(\alpha p, \alpha q) && \text{by definition of } \mathsf{d} \text{ (Definition 3.1 page 7)} \\ &\triangleq \int_0^{\phi_\alpha} \sqrt{[\mathsf{r}(\theta; \alpha p, \alpha q)]^2 + \left[\frac{\mathrm{d}\mathsf{r}(\theta; \alpha p, \alpha q)}{\mathrm{d}\theta}\right]^2}\, \mathrm{d}\theta && \text{by definition of } \mathsf{R} \text{ (Definition 3.1 page 7)} \\ &= \int_0^{\phi} \sqrt{[\alpha \mathsf{r}(\theta; p, q)]^2 + \left[\frac{\mathrm{d}}{\mathrm{d}\theta} \alpha \mathsf{r}(\theta; p, q)\right]^2}\, \mathrm{d}\theta && \text{by item (3b)} \\ &= |\alpha| \int_0^{\phi} \sqrt{[\mathsf{r}(\theta; p, q)]^2 + \left[\frac{\mathrm{d}}{\mathrm{d}\theta} \mathsf{r}(\theta; p, q)\right]^2}\, \mathrm{d}\theta && \text{by } \textit{linearity} \text{ of } \int_0^\phi \mathrm{d}\theta \text{ operator} \\ &\triangleq |\alpha| \mathsf{R}(p, q) && \text{by definition of } \mathsf{R} \text{ (Definition 3.1 page 7)} \end{aligned}$$

(4) Proof that d does *not* induce a norm on X: This follows directly from item (2) and Theorem D.14 (page 30).

(5) Proof that *balls* (Definition A.5 page 22) in d are in general *not convex* (Definition D.2 page 26):
This is demonstrated graphically in Figure 2 (page 19) and Figure 3 (page 20).
For an algebraic demonstration, consider the following:[24]

(a) Let $\mathsf{B}((0,1), 1)$ be the *unit ball* in $(\mathbb{R}^2, \mathsf{d})$ centered at $(0, 1)$.

(b) Let $p \triangleq (-0.70, -1.12)$, $q \triangleq (0.70, -1.12)$, $r \triangleq (0, -1.12)$, and $\lambda = 1/2$.

(c) Then

$$\begin{aligned} \mathsf{d}((0,1), p) &= 0.959536\cdots < 1 &&\implies p \in \mathsf{B}((0,1),1) \quad \text{and} \\ \mathsf{d}((0,1), q) &= 0.959536\cdots < 1 &&\implies q \in \mathsf{B}((0,1),1) \quad \text{BUT} \\ \mathsf{d}((0,1), r) &= 1.060688\cdots > 1 &&\implies r \notin \mathsf{B}((0,1),1) \end{aligned}$$

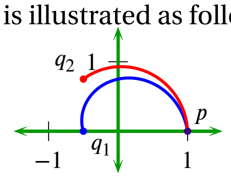
[22] 💾 See experiment log file "lab_larc_distances_R2.xlg" generated by the program "larc.exe".
[23] 💾 See experiment log file "lab_larc_distances_R2.xlg" generated by the program "larc.exe".
[24] 💾 See experiment log file "lab_larc_distances_R2.xlg" generated by the program "larc.exe".

(d) This implies that the set $\mathsf{B}((0,1),1)$ is *not convex* because

$$\begin{aligned}
\lambda p + (1-\lambda)q &\triangleq \frac{1}{2}(-0.70,-1.12) + \left(1-\frac{1}{2}\right)(0.70,-1.12) && \text{by item (5b)}\\
&= (0,-1.12)\\
&\triangleq r && \text{by item (5b)}\\
&\notin \mathsf{B}((0,1),1) && \text{by item (5c)}\\
&\implies \text{the set } \mathsf{B}((0,1),1) \text{ is } \textit{not convex} && \text{by Definition D.2 page 26}
\end{aligned}$$

✏

Remark 3.14 (Lagrange arc distance versus Euclidean metric) As is implied by the metric balls illustrated in Figure 2 (page 19) and Figure 3 (page 20), the *Lagrange arc distance* d and *Euclidean metric* p are similar in the sense that they often lead to the same results[25] in determining which of the two points q_1 or q_2 is "closer" to a point p. But in some cases the two metrics lead to two different results. One such case is illustrated as follows:[26]



$$\begin{aligned}
\mathsf{d}(p,q_1) &\triangleq \mathsf{d}((1,0),(-0.5,0)) &&= 0.767324\cdots\\
\mathsf{d}(p,q_2) &\triangleq \mathsf{d}((1,0),(-0.5,0.75)) &&= 0.654039\cdots\\
\mathsf{p}(p,q_1) &\triangleq \mathsf{p}((1,0),(-0.5,0)) &&= 1.5\\
\mathsf{p}(p,q_2) &\triangleq \mathsf{p}((1,0),(-0.5,0.75)) &&= \sqrt{(1.5)^2+(0.75)^2} = 1.677050\cdots
\end{aligned}$$

That is, q_2 is closer than q_1 to p with respect to the *Lagrange arc distance*,
but q_1 is closer than q_2 to p with respect to the *Euclidean metric*.

Remark 3.15 (Lagrange arc distance and power triangle inequality) Greenhoe (2016a) introduced a relation therein called the *power triangle inequality* with parameters (α,σ) which is satisfied if

$$\mathsf{d}(p,r) \le 2\sigma\left[\frac{1}{2}\mathsf{d}^\alpha(p,q) + \frac{1}{2}\mathsf{d}^\alpha(q,r)\right]^{\frac{1}{\alpha}} \quad \text{for all triples } (p,q,r) \text{ in the distance space.}$$

Note that the *triangle inequality* is a special case of this relation at $(\alpha,\sigma)=(1,1)$. Greenhoe (2016a) demonstrated that several of properties of metric spaces also hold in distance spaces in which the power triangle inequality holds and (α,σ) is constrained to satisfy $\sigma = \frac{1}{2}(2^{\frac{1}{\alpha}})$. Besides the triangle inequality, another special case of the power triangle inequality is the *square mean root inequality* at $(\alpha,\sigma)=\left(\frac{1}{2},2\right)$. One might be encouraged to hope that the Lagrange arc distance function satisfies this inequality because it succeeds where the triangle inequality failed in item (1) of the proof of Theorem 3.13 (page 15):

$$\begin{aligned}
\mathsf{d}(p,r) &\triangleq \mathsf{d}((1,0),(-0.5,0)) && = 0.767324\cdots\\
&\le 1.177652\cdots && = 2\sigma\left[\tfrac{1}{2}(0.692330\cdots)^\alpha + \tfrac{1}{2}(0.064076\cdots)^\alpha\right]^{\frac{1}{\alpha}}\\
&= 2\sigma\left[\tfrac{1}{2}\mathsf{d}^\alpha((1,0),(-0.5,0.2)) \right.&& \left.+ \tfrac{1}{2}\mathsf{d}^\alpha((-0.5,0.2),(-0.5,0))\right]^{\frac{1}{\alpha}}\\
&\triangleq 2\sigma\left[\tfrac{1}{2}\mathsf{d}^\alpha(p,q) + \tfrac{1}{2}\mathsf{d}^\alpha(q,r)\right]^{\frac{1}{\alpha}}\\
&\implies \textit{power triangle inequality holds} \text{ at } (p,q,r) \text{ in } (X,\mathsf{d}) \text{ with } (\alpha,\sigma)=\left(\tfrac{1}{2},2\right)
\end{aligned}$$

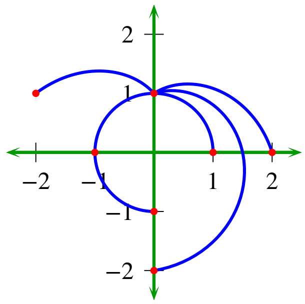
However, the *power triangle inequality* at $(\alpha,\sigma)=\left(\frac{1}{2},2\right)$ does not in general hold in this distance space as demonstrated for the triple $(p,q,r)\triangleq((1,1),(0.03,0.04),(-0.05,-0.06))$:

$$\begin{aligned}
\mathsf{d}(p,r) &\triangleq \mathsf{d}((1,1),(-0.05,-0.06)) && = 0.867534\cdots\\
&\not\le 0.832627\cdots && = 2\sigma\left[\tfrac{1}{2}(0.435860\cdots)^\alpha + \tfrac{1}{2}(0.063648\cdots)^\alpha\right]^{\frac{1}{\alpha}}\\
&= 2\sigma\left[\tfrac{1}{2}\mathsf{d}^\alpha((1,1),(0.03,0.04)) \right.&& \left.+ \tfrac{1}{2}\mathsf{d}^\alpha((0.03,0.04),(-0.05,-0.06))\right]^{\frac{1}{\alpha}}\\
&\triangleq 2\sigma\left[\tfrac{1}{2}\mathsf{d}^\alpha(p,q) + \tfrac{1}{2}\mathsf{d}^\alpha(q,r)\right]^{\frac{1}{\alpha}}\\
&\implies \textit{power triangle inequality fails} \text{ for } (\alpha,\sigma)=\left(\tfrac{1}{2},2\right)
\end{aligned}.$$

3.5 Examples

Example 3.16 (Lagrange arc distance in $\mathbb{R}^2$) Figure 1 (page 18) illustrates the *Lagrange arc distance* on some pairs of points in $\mathbb{R}^2$.[27]

[25] For empirical evidence of this, see Greenhoe (2016b).
[26] See experiment log file "lab_larc_distances_R2.xlg" generated by the program "larc.exe".
[27] See experiment log file "lab_larc_distances_R2.xlg" generated by the program "larc.exe".



$$
\begin{aligned}
\mathsf{d}((0,1),(1,0)) &= \tfrac{1}{2}\\
\mathsf{d}((0,1),(-1,0)) &= \tfrac{1}{2}\\
\mathsf{d}((0,1),(0,-1)) &= 1\\
\mathsf{d}((1,0),(0,-1)) &= \tfrac{1}{2}\\
\mathsf{d}((1,0),(-1,0)) &= 1\\
\mathsf{d}((-1,0),(0,-1)) &= \tfrac{1}{2}\\
\mathsf{d}((0,1),(2,0)) &= 0.8167968\cdots\\
\mathsf{d}((0,1),(0,-2)) &= 1.5346486\cdots\\
\mathsf{d}((0,1),(-2,1)) &= 0.6966032\cdots
\end{aligned}
$$

Figure 1: *Lagrange arc distance* examples in $\mathbb{R}^2$

$\mathsf{d}((0,1,0),(1,0,0)) = \tfrac{1}{2}$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((0,1,0),(0,0,1)) = \tfrac{1}{2}$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((0,1,0),(0,0,-1)) = \tfrac{1}{2}$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((0,1,0),(-1,0,0)) = \tfrac{1}{2}$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((0,1,0),(0,-1,0)) = 1$	$\phi = \pi$	180°
$\mathsf{d}((1,0,0),(0,0,1)) = \tfrac{1}{2}$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((1,0,0),(0,0,-1)) = \tfrac{1}{2}$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((1,0,0),(-1,0,0)) = 1$	$\phi = \pi$	180°
$\mathsf{d}((1,0,0),(0,-1,0)) = \tfrac{1}{2}$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((0,0,1),(0,0,-1)) = 1$	$\phi = \pi$	180°
$\mathsf{d}((0,0,1),(-1,0,0)) = \tfrac{1}{2}$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((0,0,1),(0,-1,0)) = \tfrac{1}{2}$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((0,0,-1),(-1,0,0)) = \tfrac{1}{2}$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((0,0,-1),(0,-1,0)) = \tfrac{1}{2}$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((-1,0,0),(0,-1,0)) = \tfrac{1}{2}$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((0,1,0),(2,0,0)) = 0.816796\cdots$	$\phi = \tfrac{\pi}{2}$	90°
$\mathsf{d}((0,1,0),(0,-2,0)) = 1.534648\cdots$	$\phi = \pi$	180°
$\mathsf{d}((0,1,0),(-2,1,0)) = 0.696603\cdots$	$\phi \approx 1.107$	63°
$\mathsf{d}((0,1,0),(-1,0,-1)) = 0.617920\cdots$	$\phi = \pi$	90°
$\mathsf{d}((1,1,1),(-\tfrac{1}{2},\tfrac{1}{4},-2)) = 1.366268\cdots$	$\phi \approx 2.2466$	128.72°

Table 1: Some examples of Lagrange arc distances in $\mathbb{R}^3$ (see Example 3.17 page 18)

Example 3.17 (Lagrange arc distance in $\mathbb{R}^3$) Some examples of Lagrange arc distances in $\mathbb{R}^3$ are given in Table 1 (page 18).[28]

Example 3.18 (Lagrange arc distance balls in $\mathbb{R}^2$) Some unit balls in $\mathbb{R}^2$ in with respect to the *Lagrange arc distance* are illustrated in Figure 2 (page 19).

Example 3.19 (Lagrange arc distance balls in $\mathbb{R}^3$) Some unit balls in $\mathbb{R}^3$ with respect to the *Lagrange arc distance* are illustrated in Figure 3 (page 20).

4 Example applications

4.1 Application to symbolic sequence processing

Example 4.1 Define the *real die metric* $\dot{\mathsf{d}}(x,y)$ as the number of edges on a real die that one must traverse to travel from face x to face y. These distances are summarized below in the table on the left. This

[28] See experiment log file "lab_larc_distances_R3.xlg" generated by the program "larc.exe".

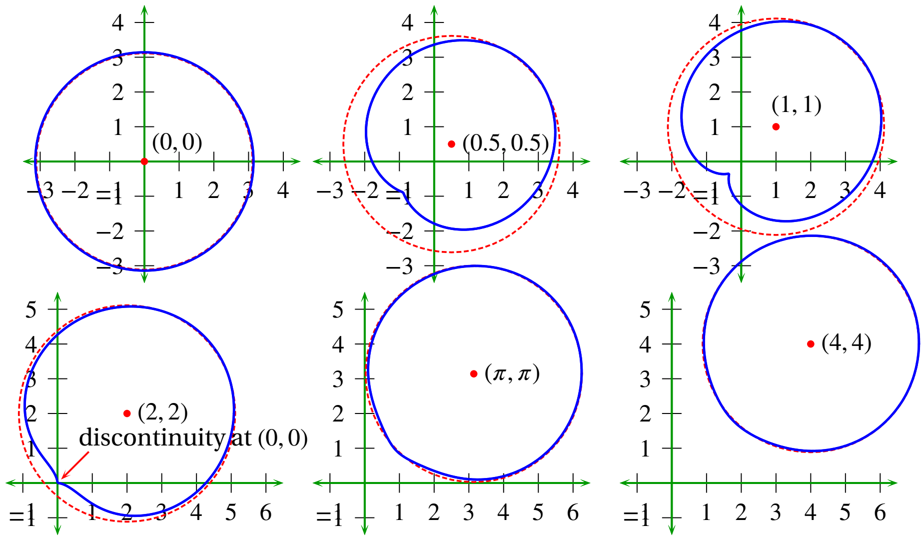


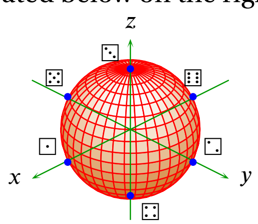
Figure 2: *Lagrange arc distance* unit balls, and dashed $\frac{1}{\pi}$-*scaled Euclidean metric* unit balls

stochastic process can be mapped through a random variable X into the structure $(\mathbb{R}^3, \leq, \mathsf{d})$ where

$$\begin{array}{llllll} \mathsf{X}(⚀) \triangleq (+1,0,0), & \mathsf{X}(⚁) \triangleq (0,+1,0), & \mathsf{X}(⚂) \triangleq (0,0,+1), \\ \mathsf{X}(⚃) \triangleq (0,0,-1), & \mathsf{X}(⚄) \triangleq (0,-1,0), & \mathsf{X}(⚅) \triangleq (-1,0,0). \end{array},$$

$\leq = \varnothing$, and d is the 2-scaled *Lagrange arc distance* d defined as follows: $\mathsf{d}(p,q) \triangleq 2\mathsf{p}(p,q)$
where p is the *Lagrange arc distance* (Definition 3.1 page 7). This is illustrated below on the right.

$\dot{\mathsf{d}}(x,y)$	⚀	⚁	⚂	⚃	⚄	⚅
⚀	0	1	1	1	1	2
⚁	1	0	1	1	2	1
⚂	1	1	0	2	1	1
⚃	1	1	2	0	1	1
⚄	1	2	1	1	0	1
⚅	2	1	1	1	1	0



Note that the *2-scaled Lagrange arc distance* d is an extension of $\dot{\mathsf{d}}$. We can also say that X is an *isometry* (Definition A.6 page 22) and that the two structures are *isometric*. For example,

$$\mathsf{d}[\mathsf{X}(⚀),\mathsf{X}(⚁)] = \mathsf{d}[(1,0,0),(0,1,0)] = 1 = \dot{\mathsf{d}}(⚀,⚁) \quad \text{and}$$
$$\mathsf{d}[\mathsf{X}(⚀),\mathsf{X}(⚅)] = \mathsf{d}[(1,0,0),(-1,0,0)] = 2 = \dot{\mathsf{d}}(⚀,⚅) \quad .$$

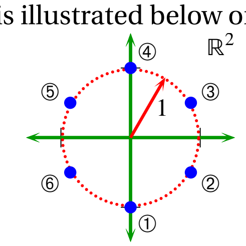
For more details, see Greenhoe (2016b) and Greenhoe (2015).

Example 4.2 Define the *spinner metric* $\dot{\mathsf{d}}(x,y)$ as the number of positions one must traverse to travel from position x to position y on a 6-position spinner. These distances are summarized below in the table on the left. This stochastic process can be mapped through a random variable X into the structure $(\mathbb{R}^2, \leq, \mathsf{d})$ where $\mathbb{C} \triangleq \mathbb{R}^2$ and

$$\begin{array}{lll} \mathsf{X}(①) \triangleq \exp\left(-90 \times \frac{\pi}{180} i\right), & \mathsf{X}(②) \triangleq \exp\left(-30 \times \frac{\pi}{180} i\right), & \mathsf{X}(③) \triangleq \exp\left(30 \times \frac{\pi}{180} i\right), \\ \mathsf{X}(④) \triangleq \exp\left(90 \times \frac{\pi}{180} i\right), & \mathsf{X}(⑤) \triangleq \exp\left(150 \times \frac{\pi}{180} i\right), & \mathsf{X}(⑥) \triangleq \exp\left(210 \times \frac{\pi}{180} i\right). \end{array},$$

$\leq = \varnothing$, and d is the 3-scaled *Lagrange arc distance* d defined as follows: $\mathsf{d}(p,q) \triangleq 3\mathsf{p}(p,q)$
where p is the *Lagrange arc distance* (Definition 3.1 page 7). This is illustrated below on the right.

$\dot{\mathsf{d}}(x,y)$	①	②	③	④	⑤	⑥
①	0	1	2	3	2	1
②	1	0	1	2	3	2
③	2	1	0	1	2	3
④	3	2	1	0	1	2
⑤	2	3	2	1	0	1
⑥	1	2	3	2	1	0



Used together with the X, the *3-scaled Lagrange arc distance* d is an extension of $\dot{\mathsf{d}}$. We can again say that

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

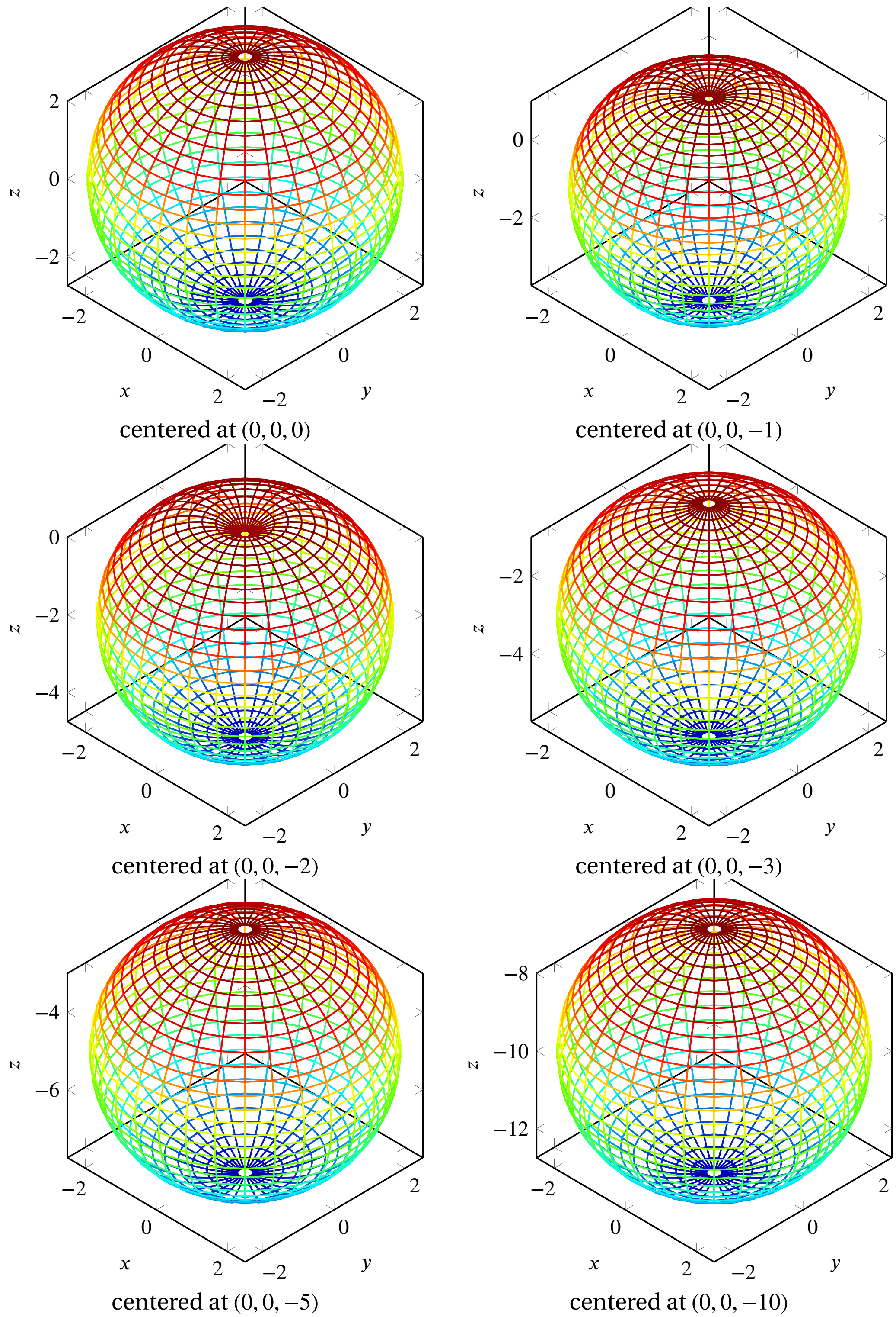


Figure 3: *unit Lagrange arc distance balls* in $\mathbb{R}^3$

X is an *isometry* and that the two structures are *isometric*. For example,

$$\begin{array}{rclclcll} \mathsf{d}\left[\mathsf{X}(①),\mathsf{X}(②)\right] &=& \mathsf{d}[(0,-1),(\sqrt{3}/2,-1/2)] &=& 1 &=& \dot{\mathsf{d}}(①,②) & \text{and}\\ \mathsf{d}\left[\mathsf{X}(①),\mathsf{X}(③)\right] &=& \mathsf{d}[(0,-1),(\sqrt{3}/2,+1/2)] &=& 2 &=& \dot{\mathsf{d}}(①,③) & \text{and}\\ \mathsf{d}\left[\mathsf{X}(①),\mathsf{X}(④)\right] &=& \mathsf{d}[(0,-1),(0,1)] &=& 3 &=& \dot{\mathsf{d}}(①,④) & . \end{array}$$

For more details, see Greenhoe (2016b) and Greenhoe (2015).

4.2 Calculations involving near earth objects

In the physical sciences, an application of extended spherical metrics is calculations involving trajectories of near earth objects, especially those that actually enter earth's atmosphere. One recent example is the *Chelyabinsk meteor*, an approximately 1000 kilogram meteorite which impacted the earth 2013 February 15 with an "approximate total impact energy" equivalent to 440 "kilotons of TNT explosives".[29]

A further application is calculations involving trajectories of man-made space vehicles launched from earth and satellites in earth orbit. Stable orbits of man-made objects vary widely from *low earth orbit* such as is used by the *International Space Station* at about 400 kilometers, to *medium earth orbit* such as is used by the *Global Positioning System* at about 20,000 kilometers, to *highly elliptical orbit* such the *Interstellar Boundary Explorer* with an perigee of approximately 56,000 kilometers and an apogee of approximately 279,000 kilometers.[30]

Appendix A Distance spaces

Definition A.1 [31] A function d in the set $\mathbb{R}^{X\times X}$ (Definition 1.8 page 3) is a **distance** if

1. $\mathsf{d}(x,y) \geq 0 \quad \forall x,y\in X$ (*non-negative*) and
2. $\mathsf{d}(x,y) = 0 \iff x=y \quad \forall x,y\in X$ (*nondegenerate*) and
3. $\mathsf{d}(x,y) = \mathsf{d}(y,x) \quad \forall x,y\in X$ (*symmetric*)

The pair (X,d) is a **distance space** if d is a *distance* on a set X.

Definition A.2 [32] Let (X,d) be a *distance space* and 2^X be the *power set* of X (Definition 1.9 page 3). The **diameter** in (X,d) of a set $A\in 2^X$ is

$$\operatorname{diam} A \triangleq \begin{cases} 0 & \text{for } A=\varnothing\\ \sup\{\mathsf{d}(x,y)\,|\,x,y\in A\} & \text{otherwise}\end{cases}$$

Definition A.3 [33] Let (X,d) be a *distance space*. Let 2^X be the *power set* (Definition 1.9 page 3) of X. A set A is **bounded** in (X,d) if $A\in 2^X$ and $\operatorname{diam} A<\infty$.

The next theorem lists five properties that do *not* in general hold in a *distance space*. Note that if a *distance space* is a *metric space*, then all five of the properties *do* hold.

Theorem A.4 [34] *Let $(x_n)_{n\in\mathbb{Z}}$ be a sequence in a distance space (X,d). The distance space (X,d) does not necessarily have all the nice properties that a metric space (Definition B.1 page 22) has. In particular, note the*

[29] Ruzicka et al. (2015) pages 8–9, Yeomans and Chodas (2013)

[30] NASA (2015), page 21, http://www.n2yo.com/satellite/?s=33401, http://www.astronautix.com/project/navstar.htm, https://directory.eoportal.org/web/eoportal/satellite-missions/i/ibex

[31] Menger (1928) page 76 ⟨"Abstand a b definiert ist…" (distance from a to b is defined as…")⟩, Wilson (1931) page 361 ⟨§1., "distance", "semi-metric space"⟩, Blumenthal (1938) page 38, Blumenthal (1953) page 7 ⟨"DEFINITION 5.1. A distance space is called semimetric provided…"⟩, Galvin and Shore (1984) page 67 ⟨"distance function"⟩, Laos (1998) page 118 ⟨"distance space"⟩, Khamsi and Kirk (2001) page 13 ⟨"semimetric space"⟩, Bessenyei and Pales (2014) page 2 ⟨"semimetric space"⟩, Deza and Deza (2014) page 3 ⟨"**distance** (or **dissimilarity**)"⟩

[32] in *metric space*: Hausdorff (1937), page 166, Copson (1968), page 23, Michel and Herget (1993), page 267, Molchanov (2005) page 389

[33] in *metric space*: Thron (1966), page 154 ⟨definition 19.5⟩, Bruckner et al. (1997) page 356

[34] Greenhoe (2016a), Heath (1961) page 810 ⟨THEOREM⟩, Galvin and Shore (1984) page 71 ⟨2.3 LEMMA⟩

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

following:

1. d *is a distance in* (X,d) $\not\Longrightarrow$ d *is continuous in* (X,d)
2. B *is an open ball in* (X,d) $\not\Longrightarrow$ B *is open in* (X,d)
3. $\boldsymbol{B}$ *is the set of all open balls in* (X,d) $\not\Longrightarrow$ $\boldsymbol{B}$ *is a base for a topology on* X
4. $(\!(x_n)\!)$ *is convergent in* (X,d) $\not\Longrightarrow$ *limit is unique*
5. $(\!(x_n)\!)$ *is convergent in* (X,d) $\not\Longrightarrow$ $(\!(x_n)\!)$ *is Cauchy in* (X,d)

Definition A.5 [35] Let (X,d) be a *distance space* (Definition A.1).
An **open ball** in (X,d) centered at x with radius r is the set $\mathsf{B}(x,r) \triangleq \{y\in X|\mathsf{d}(x,y)<r\}$.
A **closed ball** in (X,d) centered at x with radius r is the set $\overline{\mathsf{B}}(x,r) \triangleq \{y\in X|\mathsf{d}(x,y)\le r\}$.

Definition A.6 [36] Let (X,d) and (Y,p) be *distance spaces*.
The function $\mathsf{f}\in Y^X$ is an **isometry** on $(Y,\mathsf{p})^{(X,\mathsf{d})}$ if $\mathsf{d}(x,y)=\mathsf{p}(\mathsf{f}(x),\mathsf{f}(y)) \quad \forall x,y\in X$.
The spaces (X,d) and (Y,p) are **isometric** if there exists an isometry on $(Y,\mathsf{p})^{(X,\mathsf{d})}$.

Appendix B Metric spaces

Definition B.1 [37] Let X be a set and $\mathbb{R}^{\vdash}$ the set of non-negative real numbers.
A function $\mathsf{d}\in\mathbb{R}^{\vdash X\times X}$ is a **metric** on X if

1. $\mathsf{d}(x,y) \ge 0 \quad \forall x,y\in X$ (*non-negative*) and
2. $\mathsf{d}(x,y) = 0 \iff x=y \quad \forall x,y\in X$ (*nondegenerate*) and
3. $\mathsf{d}(x,y) = \mathsf{d}(y,x) \quad \forall x,y\in X$ (*symmetric*) and
4. $\mathsf{d}(x,y) \le \mathsf{d}(x,z)+\mathsf{d}(z,y) \quad \forall x,y,z\in X$ (*subadditive/ triangle inequality*).[38]

A **metric space** is the pair (X,d).

Definition B.2 [39] Let X be a set and $\mathsf{d}\in\mathbb{R}^{X\times X}$. The function d is the **discrete metric** on $\mathbb{R}^{X\times X}$ if

$$\mathsf{d}(x,y)\triangleq\left\{\begin{array}{ll}0 & \text{if } x=y\\ 1 & \text{otherwise}\end{array}\right\} \qquad \forall x,y\in X$$

Definition B.3 Let X be a set and $\mathsf{d}\in\mathbb{R}^{\mathbb{R}^N}$. The function d is the **Euclidean metric** on $\mathbb{R}^N$ if

$$\mathsf{d}\big((x_1,x_2,\cdots,x_N),(y_1,y_2,\cdots,y_N)\big)\triangleq\sqrt{\sum_{n=1}^{N}(x_n-y_n)^2} \qquad \forall (x_1,x_2,\cdots,x_N),(y_1,y_2,\cdots,y_N)\in\mathbb{R}^N$$

Appendix C Polynomial interpolation

C.1 Lagrange interpolation

Definition C.1 [40] The **Lagrange polynomial** $L_{P,n}(x)$ with respect to the $n+1$ points $P=\{(x_k,y_k)|k=0,1,2,\ldots,n\}$ is defined as

$$L_{P,n}(x)\triangleq\sum_{k=0}^{n}y_k\prod_{m\neq n}\frac{x-x_m}{x_k-x_m}$$

[35] in *metric space*: Aliprantis and Burkinshaw (1998), page 35

[36] Thron (1966), page 153 ⟨definition 19.4⟩, Giles (1987) page 124 ⟨Definition 6.22⟩, Khamsi and Kirk (2001) page 15 ⟨Definition 2.4⟩, Kubrusly (2001) page 110

[37] Dieudonné (1969), page 28, Copson (1968), page 21, Hausdorff (1937), page 109, Fréchet (1928), Fréchet (1906), page 30

[38] Euclid (circa 300BC) ⟨Book I Proposition 20⟩

[39] Busemann (1955) page 4 ⟨Comments on the axioms⟩, Giles (1987), page 13, Copson (1968), page 24, Khamsi and Kirk (2001) page 19 ⟨Example 2.1⟩

[40] Waring (1779) page 60, Euler (1783) page 165 ⟨§. 10. Problema 2. Corollarium 3.⟩, Gauss (1866), Lagrange (1877), Matthews and Fink (1992), page 206, Meijering (2002) ⟨historical background⟩

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

Proposition C.2 *Let $L_{P,n}(x)$ be the Lagrange polynomial with respect to the points $P = \{(x_k, y_k)|k = 0, 1, 2, \ldots, n\}$.*

1. *$L_{P,n}(x)$ is an nth order polynomial.*
2. *$L_{P,n}(x)$ intersects all $n+1$ points in P.*

Example C.3 (Lagrange interpolation) The Lagrange polynomial $L_{P,3}(x)$ with respect to the 4 points $P = \{(-2,1),\ (-1,3),\ (3,2),\ (5,4)\}$ is

$$L_{P,3}(x) = \frac{79}{840}x^3 + \frac{-378}{840}x^2 + \frac{-7}{840}x + \frac{2970}{840}$$

✎PROOF:

$$\begin{aligned}
L_{P,3}(x) &= \sum_{k=0}^{n} y_k \prod_{m\neq n} \frac{x - x_m}{x_k - x_m} \qquad \text{by Definition C.1} \\
&= y_0 \frac{(x+1)(x-3)(x-5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x+2)(x-3)(x-5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\
&\quad + y_2 \frac{(x+2)(x+1)(x-5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x+2)(x+1)(x-3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\
&= 1\frac{(x+1)(x-3)(x-5)}{(-2+1)(-2-3)(-2-5)} + 3\frac{(x+2)(x-3)(x-5)}{(-1+2)(-1-3)(-1-5)} \\
&\quad + 2\frac{(x+2)(x+1)(x-5)}{(3+2)(3+1)(3-5)} + 4\frac{(x+2)(x+1)(x-3)}{(5+2)(5+1)(5-3)} \\
&= 1\underbrace{\frac{x^3-7x^2+7x+15}{-35}}_{\text{roots}=-1,3,5} + 3\underbrace{\frac{x^3-6x^2-x+30}{24}}_{\text{roots}=-2,3,5} + 2\underbrace{\frac{x^3-2x^2-13x-10}{-40}}_{\text{roots}=-2,-1,5} + 4\underbrace{\frac{x^3-7x-6}{84}}_{\text{roots}=-2,-1,3} \\
&= -\frac{x^3-7x^2+7x+15}{35} + \frac{x^3-6x^2-x+30}{8} - \frac{x^3-2x^2-13x-10}{20} + \frac{x^3-7x-6}{21} \\
&= x^3\left(\frac{-8\cdot20\cdot21 + 35\cdot20\cdot21 - 35\cdot8\cdot21 + 35\cdot8\cdot20}{35\cdot8\cdot20\cdot21}\right) \\
&\quad + x^2\left(\frac{7\cdot8\cdot20\cdot21 - 6\cdot35\cdot20\cdot21 + 2\cdot35\cdot8\cdot21 + 0\cdot35\cdot8\cdot20}{35\cdot8\cdot20\cdot21}\right) \\
&\quad + x\left(\frac{-7\cdot8\cdot20\cdot21 - 35\cdot20\cdot21 + 13\cdot35\cdot8\cdot21 - 7\cdot35\cdot8\cdot20}{35\cdot8\cdot20\cdot21}\right) \\
&\quad + \left(\frac{-15\cdot8\cdot20\cdot21 + 30\cdot35\cdot20\cdot21 + 10\cdot35\cdot8\cdot21 - 6\cdot35\cdot8\cdot20}{35\cdot8\cdot20\cdot21}\right) \\
&= \frac{11060}{117600}x^3 + \frac{-52920}{117600}x^2 + \frac{-980}{117600}x + \frac{415800}{117600} \\
&= \frac{79}{840}x^3 + \frac{-378}{840}x^2 + \frac{-7}{840}x + \frac{2970}{840}
\end{aligned}$$

✎

C.2 Newton interpolation

Definition C.4 [41] The **Newton polynomial** $N_{P,n}(x)$ with respect to the $n+1$ points $P = \{(x_k, y_k)|k = 0, 1, 2, \ldots, n\}$ is defined as

$$N_{P,n}(x) \triangleq \sum_{k=0}^{n} \alpha_k \prod_{m=0}^{k} (x - x_m)$$

Proposition C.5 *Let $N_{P,n}(x)$ be the Newton polynomial with respect to the points $P = \{(x_k, y_k)|k = 0, 1, 2, \ldots, n\}$.*

1. *$N_{P,n}(x)$ is an nth order polynomial.*
2. *$N_{P,n}(x)$ intersects all $n+1$ points in P.*

[41] Newton (1711), Fraser (1919), pages 9–17 ⟨Methodus differentialis: "A photographic reproduction of the original Latin text"⟩, Fraser (1919), pages 18–25 ⟨Methodus differentialis: English translation⟩, Fraser (1919), pages 1–8 ⟨historical background and notes⟩, Meijering (2002) ⟨historical background⟩, Matthews and Fink (1992), page 220

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

Example C.6 (Newton polynomial interpolation) The Newton polynomial $N_{P,3}(x)$ with respect to the 4 points $P = \{(-2,1),\ (-1,3),\ (3,2),\ (5,4)\}$ is

$$N_{P,3}(x) = \tfrac{79}{840}x^3 + \tfrac{-378}{840}x^2 + \tfrac{-7}{840}x + \tfrac{2970}{840}$$

✎PROOF:

$$\begin{aligned}
N_{P,3}(x) &= \sum_{k=0}^{n} \alpha_k \prod_{m=1}^{k} (x - x_m) \\
&= \alpha_0 + \alpha_1(x - x_0) + \alpha_2(x - x_0)(x - x_1) + \alpha_3(x - x_0)(x - x_1)(x - x_2) \\
&= \alpha_0 + \alpha_1(x + 2) + \alpha_2(x + 2)(x + 1) + \alpha_3(x + 2)(x + 1)(x - 3) \\
&= \alpha_0 + \alpha_1(x + 2) + \alpha_2(x^2 + 3x + 2) + \alpha_3(x^3 - 7x - 6) \\
&= x^3(\alpha_3) + x^2(\alpha_2) + x(-7\alpha_3 + 3\alpha_2 + \alpha_1) + (-6\alpha_3 + 2\alpha_2 + 2\alpha_1 + \alpha_0) \\
&= \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -6 & -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) & 0 \\ 1 & (x_3 - x_0) & (x_3 - x_0)(x_3 - x_1) & (x_3 - x_0)(x_3 - x_1)(x_3 - x_2) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & (-1 + 2) & 0 & 0 \\ 1 & (3 + 2) & (3 + 2)(3 + 1) & 0 \\ 1 & (5 + 2) & (5 + 2)(5 + 1) & (5 + 2)(5 + 1)(5 - 3) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 5 & 20 & 0 \\ 1 & 7 & 42 & 84 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 5 & 20 & 0 & 0 & 0 & 1 & 0 \\ 1 & 7 & 42 & 84 & 0 & 0 & 0 & 1 \end{array}\right]
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 5 & 20 & 0 & -1 & 0 & 1 & 0 \\ 0 & 7 & 42 & 84 & -1 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 20 & 0 & 4 & -5 & 1 & 0 \\ 0 & 0 & 42 & 84 & 6 & -7 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{4} & \frac{1}{20} & 0 \\ 0 & 0 & 42 & 84 & 6 & -7 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{4} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 84 & 6 - \frac{42}{5} & -7 + \frac{42}{4} & -\frac{42}{20} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & -\frac{1}{4} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 84 & -\frac{12}{5} & \frac{14}{4} & -\frac{42}{20} & 1 \end{bmatrix}
\end{aligned}$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{4}{20} & -\frac{5}{20} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 84 & -\frac{24}{10} & \frac{35}{10} & -\frac{21}{10} & \frac{10}{10} \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{4}{20} & -\frac{5}{20} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & 1 & -\frac{24}{840} & \frac{35}{840} & -\frac{21}{840} & \frac{10}{840} \end{bmatrix}
$$

$$
\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{4}{20} & -\frac{5}{20} & \frac{1}{20} & 0 \\ -\frac{24}{840} & \frac{35}{840} & -\frac{21}{840} & \frac{10}{840} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}
= \begin{bmatrix} 1 \\ 2 \\ -\frac{9}{20} \\ \frac{79}{840} \end{bmatrix}
$$

$$
N_{P,3}(x) = \begin{bmatrix} \alpha_0 \; \alpha_1 \; \alpha_2 \; \alpha_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -6 & -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}
$$

$$
= \left[\; 1 \;\middle|\; 2 \;\middle|\; -\tfrac{9}{20} \;\middle|\; \tfrac{79}{840} \;\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -6 & -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}
$$

$$
= \left[\; 1+4-\tfrac{9}{10}-\tfrac{79}{140} \;\middle|\; 2-\tfrac{27}{20}-\tfrac{79}{120} \;\middle|\; -\tfrac{9}{20} \;\middle|\; \tfrac{79}{840} \;\right] \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}
$$

$$
= \frac{79}{840}x^3 - \frac{378}{840}x^2 - \frac{7}{840}x + \frac{2970}{840}
$$

✏

Appendix D Linear spaces

D.1 Structure

Definition D.1 [42] Let $(\mathbb{F}, +, \cdot, 0, 1)$ be a *field*. Let X be a set, let $+$ be an *operator* in X^{X^2}, and let $\otimes$ be an operator in $X^{\mathbb{F}\times X}$. The structure $\boldsymbol{\Omega} \triangleq \big(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\big)$ is a **linear space** over $(\mathbb{F}, +, \cdot, 0, 1)$ if

1. $\exists \mathbb{0} \in X$ such that $\boldsymbol{x} + \mathbb{0} = \boldsymbol{x}$ $\forall \boldsymbol{x} \in X$ (+ *identity*) *
2. $\exists \boldsymbol{y} \in X$ such that $\boldsymbol{x} + \boldsymbol{y} = \mathbb{0}$ $\forall \boldsymbol{x} \in X$ (+ *inverse*)
3. $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z} = \boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$ $\forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in X$ (+ is *associative*)
4. $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{y}+\boldsymbol{x}$ $\forall \boldsymbol{x},\boldsymbol{y} \in X$ (+ is *commutative*)
5. $1 \cdot \boldsymbol{x} = \boldsymbol{x}$ $\forall \boldsymbol{x} \in X$ ($\cdot$ *identity*)
6. $\alpha \cdot (\beta \cdot \boldsymbol{x}) = (\alpha \cdot \beta) \cdot \boldsymbol{x}$ $\forall \alpha,\beta \in S$ and $\boldsymbol{x} \in X$ ($\cdot$ *associates* with $\cdot$)
7. $\alpha \cdot (\boldsymbol{x}+\boldsymbol{y}) = (\alpha \cdot \boldsymbol{x}) + (\alpha \cdot \boldsymbol{y})$ $\forall \alpha \in S$ and $\boldsymbol{x},\boldsymbol{y} \in X$ ($\cdot$ *distributes* over +)
8. $(\alpha+\beta) \cdot \boldsymbol{x} = (\alpha \cdot \boldsymbol{x}) + (\beta \cdot \boldsymbol{x})$ $\forall \alpha,\beta \in S$ and $\boldsymbol{x} \in X$ ($\cdot$ *pseudo-distributes* over +)

[42] Kubrusly (2001) pages 40–41 ⟨Definition 2.1 and following remarks⟩, Haaser and Sullivan (1991), page 41, Halmos (1948), pages 1–2, Peano (1888a) ⟨Chapter IX⟩, Peano (1888b), pages 119–120, Banach (1922) pages 134–135

Definition D.2 [43] Let $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ be a *linear space* (Definition D.1 page 25).
A set $D \subseteq X$ is **convex** in $\boldsymbol{\Omega}$ if

$$\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in D \quad \forall \boldsymbol{x},\boldsymbol{y} \in D \quad \text{and} \quad \forall \lambda \in (0,1)$$

A set is **concave** in $\boldsymbol{\Omega}$ if it is *not convex* in $\boldsymbol{\Omega}$.

D.2 Metric Linear Spaces

Metric space stucture can be added to a linear space resulting in a *metric linear space* (next definition). One key difference between metric linear spaces and normed linear spaces is that the balls in a *normed linear space* (Definition D.6 page 27) are always *convex* (Definition D.2 page 26); this is not true for all metric linear spaces (Theorem D.12 page 28).[44]

Definition D.3 [45] Let $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d}\right)$. The tuple $\boldsymbol{\Omega}$ is a **metric linear space** if

1. if $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times})\right)$ is a *linear space* and
2. d is a *metric* in $\mathbb{R}^{\boldsymbol{X}}$ and
3. $\mathsf{d}(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+\boldsymbol{z}) = \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \quad \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in X$ (*translation invariant*)[46] and
4. $\alpha_n \to \alpha$ and $\boldsymbol{x}_n \to \boldsymbol{x} \implies \alpha_n \boldsymbol{x}_n \to \alpha \boldsymbol{x}$

Theorem D.4 [47] *Let $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d}\right)$ be a metric linear space.*

$$\underbrace{\mathsf{d}(\boldsymbol{\theta}, \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \le \lambda \mathsf{d}(\boldsymbol{\theta}, \boldsymbol{x}) + (1-\lambda)\mathsf{d}(\boldsymbol{\theta}, \boldsymbol{y})}_{\mathsf{d} \text{ is a convex function}} \implies \left\{ \begin{array}{l} \mathsf{B}(\theta, r) \in \boldsymbol{\Omega} \text{ is convex} \\ \forall \theta \in X,\, r \in \mathbb{R}^+ \end{array} \right\}$$

✎PROOF:

$$\begin{aligned}
\mathsf{d}(\boldsymbol{\theta}, \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &\le \lambda \mathsf{d}(\boldsymbol{\theta}, \boldsymbol{x}) + (1-\lambda)\mathsf{d}(\boldsymbol{\theta}, \boldsymbol{y}) && \text{by convexity hypothesis} \\
&\le \lambda r + (1-\lambda) r && \forall \boldsymbol{x}, \boldsymbol{y} \in \mathsf{B}(\theta, r) \\
&= r \\
&\implies \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in \mathsf{B}(\theta, r) && \forall \boldsymbol{x}, \boldsymbol{y} \in \mathsf{B}(\theta, r) \\
&\implies \mathsf{B}(\theta, r) \in (\boldsymbol{X}, \mathsf{d}) \text{ is convex} && \forall \theta \in \boldsymbol{X}
\end{aligned}$$

✏

Theorem D.5 [48] *Let $\left(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \mathsf{d}\right)$ be a real metric linear space.*

$$\left\{ \begin{array}{llll} 1. & \mathsf{d}(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+\boldsymbol{z}) = \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) & \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \boldsymbol{X} & \text{(translation invariant) and} \\ 2. & \mathsf{d}(\lambda\boldsymbol{x}, \lambda\boldsymbol{y}) = \lambda\, \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) & \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X},\ \lambda \in [0,1] & \text{(homogeneous)} \end{array} \right\}$$
$$\implies \{\mathsf{B}(\theta, r) \in (\boldsymbol{X}, \mathsf{d}) \quad \text{is convex} \quad \forall \theta \in \boldsymbol{X}, r \in \mathbb{R}^+\}$$

---

[43] Mitrinović et al. (2010) page 1, van de Vel (1993) pages 5–6, Bollobás (1999), page 2
[44] Bruckner et al. (1997) page 478
[45] Maddox (1989) page 90, Bruckner et al. (1997) page 477 ⟨Definition 12.3⟩, Rolewicz (1985) page 1, Loève (1977) page 79
[46] Some authors do not require the *translation invariant* property for the definition of the *metric linear space*, as indicated by the following references: Maddox (1989) page 90 ⟨"Some authors…do not include translation invariance in the definition of metric linear space, since they use a theorem of Kakutani to show that a non-translation invariant metric may be replaced by a translation invariant metric which yields the same topolgy."⟩, Friedman (1970) page 125 ⟨Definition 4.1.4⟩, Dobrowolski and Mogilski (1995) page 86
[47] Norfolk (1991), page 5
[48] Norfolk (1991) pages 5–6, `http://groups.google.com/group/sci.math/msg/a6f0a7924027957d`

✎PROOF:

$$
\begin{aligned}
&\mathsf{d}(\boldsymbol{\theta}, \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \\
&= \mathsf{d}(\mathbb{0}, \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} - \boldsymbol{\theta}) && \text{by translation invariance hypothesis} \\
&= \mathsf{d}(\mathbb{0}, \lambda(\boldsymbol{x} - \boldsymbol{\theta}) + (1-\lambda)(\boldsymbol{y} - \boldsymbol{\theta})) \\
&\le \mathsf{d}(\mathbb{0}, \lambda(\boldsymbol{x} - \boldsymbol{\theta})) + \mathsf{d}(\lambda(\boldsymbol{x} - \boldsymbol{\theta}), \lambda(\boldsymbol{x} - \boldsymbol{\theta}) + (1-\lambda)(\boldsymbol{y} - \boldsymbol{\theta})) && \text{by } \textit{subadditive} \text{ property} \\
&= \mathsf{d}(\mathbb{0}, \lambda(\boldsymbol{x} - \boldsymbol{\theta})) + \mathsf{d}(\mathbb{0}, \mathbb{0} + (1-\lambda)(\boldsymbol{y} - \boldsymbol{\theta})) && \text{by translation invariance hypothesis} \\
&= \lambda\mathsf{d}(\mathbb{0}, \boldsymbol{x} - \boldsymbol{\theta}) + (1-\lambda)\mathsf{d}(\mathbb{0}, \boldsymbol{y} - \boldsymbol{\theta}) && \text{by homogeneous hypothesis} \\
&= \lambda\mathsf{d}(\boldsymbol{\theta}, \boldsymbol{x}) + (1-\lambda)\mathsf{d}(\boldsymbol{\theta}, \boldsymbol{y}) && \text{by translation invariance hypothesis} \\
&\le \lambda r + (1-\lambda) r && \forall \boldsymbol{x}, \boldsymbol{y} \in \mathsf{B}(\theta, r) \\
&= r \\
\Longrightarrow\; & \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in \mathsf{B}(\theta, r) && \forall \boldsymbol{x}, \boldsymbol{y} \in \mathsf{B}(\theta, r) \\
\Longrightarrow\; & \mathsf{B}(\theta, r) \in (\boldsymbol{X}, \mathsf{d}) \text{ is convex} && \forall \theta \in \boldsymbol{X}
\end{aligned}
$$

✏

D.3 Normed Linear Spaces

Definition D.6 [49] Let $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ be a *linear space* (Definition D.1 page 25) and $|\cdot| \in \mathbb{R}^{\mathbb{F}}$ the *absolute value* function. A functional $\|\cdot\|$ in $\mathbb{R}^{\boldsymbol{X}}$ is a **norm** if

1. $\|\boldsymbol{x}\| \ge 0 \quad \forall \boldsymbol{x} \in X$ (*strictly positive*) and
2. $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \mathbb{0} \quad \forall \boldsymbol{x} \in X$ (*nondegenerate*) and
3. $\|\alpha\boldsymbol{x}\| = |\alpha|\,\|\boldsymbol{x}\| \quad \forall \boldsymbol{x} \in X,\ \alpha \in \mathbb{C}$ (*homogeneous*) and
4. $\|\boldsymbol{x} + \boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\| \quad \forall \boldsymbol{x}, \boldsymbol{y} \in X$ (*subadditive/ triangle inquality*).

A **normed linear space** is the tuple $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$.

Example D.7 (The usual norm) [50] Let $\mathbb{R}^{\mathbb{R}}$ be the set of all functions with domain and range the set of *real numbers* $\mathbb{R}$.

The **absolute value** $|\cdot| \in \mathbb{R}^{\mathbb{R}}$ is a *norm*.

Example D.8 (l_p norms) Let $(\!(x_n)\!)_{n\in\mathbb{Z}}$ be a *sequence* of real numbers.

$$\left\|(\!(x_n)\!)\right\|_p \triangleq \left(\sum_{n\in\mathbb{Z}} |x_n|^p\right)^{\frac{1}{p}} \qquad \text{is a norm for } p \in [1 : \infty]$$

D.4 Relationship between metrics and norms

D.4.1 Metrics generated by norms

Theorem D.9 [51] *Let* $\mathsf{d} \in \mathbb{R}^{\boldsymbol{X}\times\boldsymbol{X}}$ *be a function on a real normed linear space* $(X, +, \cdot, (\mathbb{R}, \dot{+}, \dot{\times}), \|\cdot\|)$. *Let* $\mathsf{B}(\boldsymbol{x}, r) \triangleq \{\boldsymbol{y} \in \boldsymbol{X} \mid \|\boldsymbol{y} - \boldsymbol{x}\| < r\}$ *be the open ball (Definition A.5 page 22) of radius r centered at a point $\boldsymbol{x}$.*

$\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \|\boldsymbol{x} - \boldsymbol{y}\|$ *is a metric on* $\boldsymbol{X}$

The next definition defines this metric formally.

Definition D.10 [52] Let $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|)$ be a *normed linear space* (Definition D.6 page 27).
The **metric induced by the norm** $\|\cdot\|$ is the function $\mathsf{d} \in \mathbb{R}^{\boldsymbol{X}}$ such that $\mathsf{d}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \|\boldsymbol{x} - \boldsymbol{y}\| \ \forall \boldsymbol{x}, \boldsymbol{y} \in X$.

[49] Aliprantis and Burkinshaw (1998), pages 217–218, Banach (1932a), page 53, Banach (1932b), page 33, Banach (1922) page 135
[50] Giles (1987) page 3
[51] Michel and Herget (1993), page 344, Banach (1932a) page 53
[52] Giles (2000) page 1 ⟨1.1 Definition⟩

Corollary D.11 [53] *Let* $\boldsymbol{\Omega} \triangleq \left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \|\cdot\|\right)$ *be a normed linear space* (Definition D.6 page 27).
The norm $\|\cdot\|$ *is continuous in* $\boldsymbol{\Omega}$.

Theorem D.12 (next) demonstrates that **all open or closed** balls in **any normed linear space** are *convex*. However, the converse is not true—that is, a metric not generated by a norm may still produce a ball that is *convex*.

Theorem D.12 [54] *Let* $\left(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d}\right)$ *be a metric linear space* (Definition D.3 page 26). *Let* B *be an open ball* (Definition A.5 page 22).

$$\left\{\begin{array}{l}\exists \|\cdot\| \in \mathbb{R}^{\boldsymbol{X}} \text{ such that} \\ \underbrace{\mathsf{d}(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{y}-\boldsymbol{x}\|}_{\mathsf{d} \text{ is generated by a norm}}\end{array}\right\} \implies \left\{\begin{array}{lll}1. & \mathsf{B}(\boldsymbol{x},r) = \boldsymbol{x} + B(0,r) & \text{and} \\ 2. & \mathsf{B}(\mathbb{0},r) = r\,\mathsf{B}(\mathbb{0},1) & \text{and} \\ 3. & \mathsf{B}(\boldsymbol{x},r) \text{ is convex} & \text{and} \\ 4. & \boldsymbol{x} \in \mathsf{B}(\mathbb{0},r) \iff -\boldsymbol{x} \in \mathsf{B}(\mathbb{0},r) & \text{(symmetric)}\end{array}\right.$$

✎PROOF:

(1) Proof that $\mathsf{d}(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+vz) = \mathsf{d}(\boldsymbol{x},\boldsymbol{y})$ (invariant):

$$\begin{aligned}\mathsf{d}(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+vz) &= \|(\boldsymbol{y}+vz)-(\boldsymbol{x}+\boldsymbol{z})\| && \text{by left hypothesis} \\ &= \|\boldsymbol{y}-\boldsymbol{x}\| \\ &= \mathsf{d}(\boldsymbol{x},\boldsymbol{y}) && \text{by left hypothesis}\end{aligned}$$

(2) Proof that $\mathsf{B}(\boldsymbol{x},r) = \boldsymbol{x} + B(0,r)$:

$$\begin{aligned}\mathsf{B}(\boldsymbol{x},r) &= \{\boldsymbol{y} \in \boldsymbol{X} | \mathsf{d}(\boldsymbol{x},\boldsymbol{y}) < r\} && \text{by definition of open ball } B \\ &= \{\boldsymbol{y} \in \boldsymbol{X} | \mathsf{d}(\boldsymbol{y}-\boldsymbol{y}, \boldsymbol{y}-\boldsymbol{x}) < r\} && \text{by right result 1.} \\ &= \{\boldsymbol{y} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{y}-\boldsymbol{x}) < r\} \\ &= \{\boldsymbol{u}+\boldsymbol{x} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < r\} && \text{let } \boldsymbol{u} \triangleq \boldsymbol{y}-\boldsymbol{x} \\ &= \boldsymbol{x} + \{\boldsymbol{u} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < r\} \\ &= \boldsymbol{x} + B(0,r) && \text{by definition of open ball } B\end{aligned}$$

(3) Proof that $\mathsf{B}(\mathbb{0},r) = r\,\mathsf{B}(\mathbb{0},1)$:

$$\begin{aligned}\mathsf{B}(\mathbb{0},r) &= \{\boldsymbol{y} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0},\boldsymbol{y}) < r\} && \text{by definition of open ball } B \\ &= \left\{\boldsymbol{y} \in \boldsymbol{X} | \frac{1}{r}\mathsf{d}(\mathbb{0},\boldsymbol{y}) < 1\right\} \\ &= \left\{\boldsymbol{y} \in \boldsymbol{X} | \frac{1}{r}\|\boldsymbol{y}-\mathbb{0}\| < 1\right\} && \text{by left hypothesis} \\ &= \left\{\boldsymbol{y} \in \boldsymbol{X} | \left\|\frac{1}{r}\boldsymbol{y}-\frac{1}{r}\mathbb{0}\right\| < 1\right\} && \text{by homogeneous property of } \|\cdot\| \text{ page 27} \\ &= \left\{\boldsymbol{y} \in \boldsymbol{X} | \mathsf{d}\left(\frac{1}{r}\mathbb{0}, \frac{1}{r}\boldsymbol{y}\right) < 1\right\} && \text{by left hypothesis} \\ &= \{r\boldsymbol{u} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0}, \boldsymbol{u}) < 1\} && \text{let } \boldsymbol{u} \triangleq \frac{1}{r}\boldsymbol{y} \\ &= r\,\{\boldsymbol{u} \in \boldsymbol{X} | \mathsf{d}(\mathbb{0},\boldsymbol{u}) < 1\} \\ &= r\,B(\mathbb{0},\,1) && \text{by definition of open ball } B\end{aligned}$$

(4) Proof that $\mathsf{B}(\boldsymbol{p},r)$ is convex:
We must prove that for any pair of points $\boldsymbol{x}$ and $\boldsymbol{y}$ in the open ball $\mathsf{B}(\boldsymbol{p},r)$, any point $\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}$ is also in the

[53] Giles (2000) page 2
[54] Giles (2000) page 2 ⟨1.2 Remarks⟩, Giles (1987) pages 22–26 ⟨2.4 Theorem, 2.11 Theorem⟩

open ball. That is, the distance from any point $\lambda x + (1-\lambda)y$ to the ball's center p must be less than r.

$$\begin{aligned}
\mathsf{d}(p, \lambda x + (1-\lambda)y) &= \|p - \lambda x - (1-\lambda)y\| && \text{by left hypothesis} \\
&= \Big\|\underbrace{\lambda p + (1-\lambda)p}_{p} - \lambda x - (1-\lambda)y\Big\| \\
&= \|\lambda p - \lambda x + (1-\lambda)p - (1-\lambda)y\| \\
&\le \|\lambda p - \lambda x\| + \|(1-\lambda)p - (1-\lambda)y\| && \text{by subadditivity property of } \|\cdot\| \text{ page 27} \\
&= |\lambda|\,\|p - x\| + |1-\lambda|\,\|p - y\| && \text{by homogeneous property of } \|\cdot\| \text{ page 27} \\
&= \lambda\,\|p - x\| + (1-\lambda)\,\|p - y\| && \text{because } 0 \le \lambda \le 1 \\
&\le \lambda r + (1-\lambda)r && \text{because } x, y \text{ are in the ball } \mathsf{B}(p, r) \\
&= r
\end{aligned}$$

(5) Proof that $x \in \mathsf{B}(\mathbb{0}, r) \iff -x \in \mathsf{B}(\mathbb{0}, r)$ (symmetric):

$$\begin{aligned}
x \in \mathsf{B}(\mathbb{0}, r) &\iff x \in \{y \in X \,|\, \mathsf{d}(\mathbb{0}, y) < r\} && \text{by definition of open ball } B \\
&\iff x \in \{y \in X \,|\, \|y - \mathbb{0}\| < r\} && \text{by left hypothesis} \\
&\iff x \in \{y \in X \,|\, \|y\| < r\} \\
&\iff x \in \{y \in X \,|\, \|(-1)(-y)\| < r\} \\
&\iff x \in \{y \in X \,|\, |-1|\,\|-y\| < r\} && \text{by homogeneous property of } \|\cdot\| \text{ page 27} \\
&\iff x \in \{y \in X \,|\, \|-y - \mathbb{0}\| < r\} \\
&\iff x \in \{y \in X \,|\, \mathsf{d}(\mathbb{0}, -y) < r\} && \text{by left hypothesis} \\
&\iff x \in \{-u \in X \,|\, \mathsf{d}(\mathbb{0}, u) < r\} && \text{let } u \triangleq -y \\
&\iff x \in (-\{u \in X \,|\, \mathsf{d}(\mathbb{0}, u) < r\}) \\
&\iff x \in (-\mathsf{B}(\mathbb{0}, r)) \\
&\iff -x \in \mathsf{B}(\mathbb{0}, r)
\end{aligned}$$

✏

Theorem D.12 (page 28) demonstrates that if a metric d in a metric space $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d})$ is generated by a norm, then the ball $\mathsf{B}(x, r)$ in that metric linear space is *convex*. However, the converse is not true. That is, it is possible for the balls in a metric space (Y, p) to be *convex*, but yet the metric p not be generated by a norm.

D.4.2 Norms generated by metrics

Every normed linear space is also a metric linear space (Theorem D.9 page 27). However, the converse is not true—not every metric linear space is a *normed linear space*. A characterization of metric linear spaces that *are* normed linear spaces is provided by Theorem D.14 (page 30).

Lemma D.13 [55] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}), \mathsf{d})$ *be a metric linear space. Let* $\|x\| \triangleq \mathsf{d}(x, \mathbb{0})\ \forall x \in X$.

$$\underbrace{\mathsf{d}(x + z, y + z) = \mathsf{d}(x, y) \quad \forall x,y,z \in X}_{\text{translation invariant}} \implies \left\{\begin{array}{llcll}
1. & \|x\| & = & \|-x\| & \forall x \in X \quad \text{and} \\
2. & \|x\| & = & 0 \iff x = 0 & \forall x \in X \quad \text{and} \\
3. & \|x + y\| & \le & \|x\| + \|y\| & \forall x,y \in X
\end{array}\right.$$

✎PROOF:

1. Proof that $\|x\| = \|-x\|$:

$$\begin{aligned}
\|x\| &= \mathsf{d}(x, \mathbb{0}) && \text{by definition of } \|\cdot\| \\
&= \mathsf{d}(x - x, \mathbb{0} - x) && \text{by translation invariance hypothesis} \\
&= \mathsf{d}(\mathbb{0}, -x) \\
&= \|-x\| && \text{by definition of } \|\cdot\|
\end{aligned}$$

[55] Oikhberg and Rosenthal (2007) page 599

2a. Proof that $\|x\| = 0 \implies x = 0$:

$$\begin{aligned} 0 &= \|x\| && \text{by left hypothesis} \\ &= \mathsf{d}(x, \mathbb{0}) && \text{by definition of } \|\cdot\| \\ &= \mathsf{d}(x, \mathbb{0}) && \text{by definition of } \|\cdot\| \\ &\implies x = \mathbb{0} && \text{by property of metrics} \end{aligned}$$

2b. Proof that $\|x\| = 0 \impliedby x = 0$:

$$\begin{aligned} \|x\| &= \mathsf{d}(x, \mathbb{0}) && \text{by definition of } \|\cdot\| \\ &= \mathsf{d}(\mathbb{0}, \mathbb{0}) && \text{by right hypothesis} \\ &= 0 && \text{by property of metrics} \end{aligned}$$

3. Proof that $\|x + y\| \le \|x\| + \|y\|$:

$$\begin{aligned} \|x + y\| &= \mathsf{d}(x + y, \mathbb{0}) && \text{by definition of } \|\cdot\| \\ &= \mathsf{d}(x + y - y, \mathbb{0} - y) && \text{by translation invariance hypothesis} \\ &= \mathsf{d}(x, -y) \\ &\le \mathsf{d}(x, \mathbb{0}) + \mathsf{d}(\mathbb{0}, y) && \text{by property of metrics} \\ &= \mathsf{d}(x, \mathbb{0}) + \mathsf{d}(y, \mathbb{0}) && \text{by property of metrics} \\ &= \|x\| + \|y\| && \text{by definition of } \|\cdot\| \end{aligned}$$

✏

Theorem D.14 [56] *Let* $(X, +, \cdot, (\mathbb{F}, \dot{+}, \dot{\times}))$ *be a linear space. Let* $\mathsf{d}(x, y) \triangleq \|x - y\| \; \forall x, y \in X$.

$$\left.\begin{array}{llll} 1. & \mathsf{d}(x + z, y + z) = \mathsf{d}(x, y) & \forall x,y,z \in X & \text{(translation invariant) and} \\ 2. & \mathsf{d}(\alpha x, \alpha y) = |\alpha| \mathsf{d}(x, y) & \forall x,y \in X, \alpha \in \mathbb{F} & \text{(homogeneous)} \end{array}\right\} \iff \|\cdot\| \text{ is a norm}$$

✎PROOF:

(1) Proof of $\implies$ assertion:

(a) Proof that $\|\cdot\|$ is *strictly positive*: This follows directly from the definition of d.
(b) Proof that $\|\cdot\|$ is *nondegenerate*: This follows directly from Lemma D.13 (page 29).
(c) Proof that $\|\cdot\|$ is *homogeneous*: This follows from the second left hypothesis.
(d) Proof that $\|\cdot\|$ satisfies the *triangle-inequality*: This follows directly from Lemma D.13 (page 29).

(2) Proof of $\impliedby$ assertion:

$$\begin{aligned} \mathsf{d}(x + z, y + z) &= \|(x + z) - (y + z)\| && \text{by definition of d} \\ &= \|x - y\| \\ &= \mathsf{d}(x, y) && \text{by definition of d} \\ \mathsf{d}(\alpha x, \alpha y) &= \|(\alpha x) - (\alpha y)\| && \text{by definition of d} \\ &= \|\alpha(x - y)\| \\ &= |\alpha| \, \|x - y\| && \text{by definition of } \|\cdot\| \text{ page 27} \\ &= |\alpha| \mathsf{d}(x, y) && \text{by definition of d} \end{aligned}$$

✏

Appendix E C++ source code support

This paper seeks to conform to the principles of *Reproducible Research* as detailed at
http://reproducibleresearch.net/

[56] Bollobás (1999), page 21

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

This section contains a partial C++ source code listing for larc.exe, written by the author of this paper. Although "most" of this paper does not rely on any kind of computer algorithm, the author has not yet found any closed form solution for $\left\{q \in \mathbb{R}^N | \mathsf{R}(p,q) = 1\right\}$ ...and as such, it is difficult to plot Lagrange arc distance balls (as in Example 3.16 and Example 3.17 page 18) without computer support. The C++ source may help others who might be interested in the material presented in this paper. The complete and downloadable source code for larc.exe will hopefully accompany any online version of this paper.

E.1 R^2 linear space structure

```
/*==========================================================================
 * Daniel J. Greenhoe
 *==========================================================================*/
/*-------------------------------------------------------------------------
 * ordered pair (x,y) on R^2
 *-------------------------------------------------------------------------*/
class opair {
  private:
    double x,y;
  public:
    opair(double u, double v){x=u;y=v;}       //constructor using 2 long float arguments
    opair(double u){x=u;y=u;}                 //constructor using 1 long float argument
    opair(void){x=0; y=0;}                    //constructor using no arguments (set to 0,0)
    void clear(void){x=0; y=0;} //set (x,y)=(u,v)
    void  put(double u,double v){x=u; y=v;} //set (x,y)=(u,v)
    opair get(void){opair p(getx(),gety()); return p;}
    double getx(void){return x;}             //get component x
    double gety(void){return y;}             //get component y
    void   list(const char* str1, const char *str2, FILE *ptr);
    void   list(const char* str1, const char *str2){list(str1,str2,NULL);}
    void   list(FILE *fptr){list("","",fptr);}  //list contents of sequence
    void   list(void){list("","",NULL);}      //list contents of sequence
  };

/*-------------------------------------------------------------------------
 * vector | x | over R^2
 *        | y |
 * vectors on R^2 are ordered pairs
 * (and hence inherit all the properties of class opair)
 * but also have additional linear space (vector space) properties
 *-------------------------------------------------------------------------*/
class vectR2: public opair {
  public:
    vectR2(double u, double v) : opair(u,v){};       //constructor using 2 long float arguments
    vectR2(double u) : opair(u){};                  //constructor using 1 long float argument
    vectR2(void) : opair(){};                      //constructor using no arguments (set to 0,0)
    double mag(void){return sqrt(getx()*getx()+gety()*gety());}
    double norm(void){return mag();}
    double theta(void);                          //polar rotation coordinate of opair in R^2
    void polartoxy(double r, double theta){put(r*cos(theta),r*sin(theta));}
    void   add(double u, double v){put(getx()+u, gety()+v);} //p=p+q
    void   operator+=(vectR2 q){put(getx()+q.getx(), gety()+q.gety());} //p=p+q
    void   operator-=(vectR2 q){put(getx()-q.getx(), gety()-q.gety());} //p=p-q
    void   operator&=(double phi);       //rotate p by <phi> radians in R^2 plane
    void   operator*=(double a){put(a*getx(), a*gety());}
    vectR2 operator* (double a){vectR2 u(a*getx(), a*gety()); return u;}
  };

vectR2  operator-(vectR2 p);                  // -p
vectR2  operator+(vectR2 p, vectR2 q);    // p+q
vectR2  operator-(vectR2 p, vectR2 q);    // p-q
vectR2  operator&(vectR2 p,double phi);    // <p> rotated counter-clockwise by <phi>
double  operator^(vectR2 p,vectR2 q){return p.getx()*q.getx() + p.gety()*q.gety();}    // "dot
          product" of p and q

/*-------------------------------------------------------------------------
 * class of sequences over R^2
 *-------------------------------------------------------------------------*/
class seqR2 {
  private:
    long N;
    vectR2 *xy;
  public:
```


PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

```
    seqR2(long M);              //constructor
    seqR2(long M, double u);    //constructor
    void    clear(void);               //fill seqR1 with the value 0
    void    fill(double u);            //fill seqR1 with the value <u>
    void    inc(double x0, double y0,double dx, double dy);
    int     put(long n, vectR2 xy);      //put a value <u> at location n in seq.
    int     put(long n, double u, double v);
    vectR2  get(long n);               //get a value from x at location n
    double  getx(long n);              //get a value from x at location n
    double  gety(long n);              //get a value from x at location n
    long    getN(void){return N;}      //get N
    double  norm(long n);              //norm of element x_n
    void  list(const long start, const long end, const char *str1, const char *str2, FILE *ptr);
    void  list(const long start, const long end, const char *str1, const char *str2, int display,
          FILE *fptr){
        if(display)   list(start,end,str1,str2,stdout);
                      list(start,end,str1,str2,fptr);
        }
    void  list(const char* str1, const char *str2,int display,FILE *fptr){
        if(display)   list(0,N-1,str1,str2,stdout);
                      list(0,N-1,str1,str2,fptr);
        }
    void  list(const long start, const long end){list(start,end,"","",stdout);}
    void  list(void){list(0,N-1,"","",stdout);}
    void    list1(void);               //list contents of seq. using 1 space each
    void    list1(long start, long end);//
    void    test(void);
    vectR2 max(int verbose);
    vectR2 max(void){return max(0);}     //max mode=0=no print
  };

/*====================================
 * functions
 *====================================*/
extern double pqtheta(const vectR2 p, const vectR2 q); //return radians between vectors induced by
        p and q in R^2
double chordlength(vectR2 p, vectR2 q){
  vectR2  pqd=p-q; // differnce of p and q
  return pqd.norm(); // "length" of difference
  }
```

```
/*============================================================================
 * Daniel J. Greenhoe
 * normed linear space R^2
 *============================================================================*/
/*====================================
 * headers
 *====================================*/
#include<stdio.h>
#include<stdlib.h>
#include<string.h>
#include<math.h>
#include<main.h>
#include<r1.h>
#include<r2.h>
#include<r2op.h>

/*-------------------------------------------------------------------------
 * list value of opair
 *-------------------------------------------------------------------------*/
void opair::list(const char *str1, const char *str2, FILE *ptr){
  if(strlen(str1)>0)fprintf(ptr,"%s",str1);
  fprintf(ptr,"(%9.6lf,%9.6lf)",getx(),gety());
  if(strlen(str2)>0)fprintf(ptr,"%s",str2);
  }

/*-------------------------------------------------------------------------
 * polar rotation coordinate <theta> of opair point (x,y)
 * return value is in the half open interval [0:2pi)
 * return -1 on error
 *-------------------------------------------------------------------------*/
double vectR2::theta(void){
  double x=getx();
  double y=gety();
  if(x==0){
    if(y==0)      return  -1;
```


PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

```
    else if(y>0) return +PI/2;
    else         return 3*PI/2;
    }
  if(y==0){
    if(x>0)      return 0;
    else         return PI;
    }
  if(x>0&&y>0) return      atan( y/x); // 1st quadrant
  if(x>0&&y<0) return 2*PI-atan(-y/x); // 4th quadrant
  if(x<0&&y>0) return   PI-atan(-y/x); // 2nd quadrant
  if(x<0&&y<0) return   PI+atan( y/x); // 3rd quadrant
  else         return -1;
  }

/*-------------------------------------------------------------------------
 * operator: rotate (x,y) counter-clockwise by <phi> radians
 *-------------------------------------------------------------------------*/
void vectR2::operator&=(double phi){
  vectR2  p(getx(),gety());
  p = p & phi;
  put(p.getx(),p.gety());
  }

/*=====================================
 * seqR2
 *=====================================*/
/*-------------------------------------------------------------------------
 * constructor initializing seqR1 to 0
 *-------------------------------------------------------------------------*/
seqR2::seqR2(long M){
  long n;
  N=M;
  xy = (vectR2 *)malloc(N*sizeof(vectR2));
  for(n=0; n<N; n++)xy[n].clear();
  }

/*-------------------------------------------------------------------------
 * constructor initializing seqR1 to <u>
 *-------------------------------------------------------------------------*/
seqR2::seqR2(long M, double u){
  long n;
  N=M;
  xy = (vectR2 *)malloc(N*sizeof(vectR2));
  for(n=0; n<N; n++){
    xy[n].put(u,u);
    }
  }

/*-------------------------------------------------------------------------
 * fill the seqR1 with a value 0
 *-------------------------------------------------------------------------*/
void seqR2::clear(void){
  long n;
  for(n=0; n<N; n++)xy[n].clear();
  }

/*-------------------------------------------------------------------------
 * fill the seqR1 with a value <u>
 *-------------------------------------------------------------------------*/
void seqR2::fill(double u){
  long n;
  for(n=0; n<N; n++)xy[n].put(u,u);
  }

/*-------------------------------------------------------------------------
 * fill the seqR1 with (x_0, x_1, x_2, ...)
 * where x_n = x_{n-1} + (dx,dy)
 *-------------------------------------------------------------------------*/
void seqR2::inc(double x0, double y0,double dx, double dy){
  long n;
  for(n=0;n<N;n++){
    xy[n].put(x0,y0);
    x0+=dx;
    y0+=dy;
    }
  }
```

```
/*------------------------------------------------------------------------
 * put a single value <u> into the seqR1 x at location n
 *------------------------------------------------------------------------*/
int seqR2::put(long n, double u, double v){
  int retval=0;
  if(n>=N){
    fprintf(stderr,"n=%ld larger than seqR1 size N=%ld\n",n,N);
    retval=-1;
    }
  else xy[n].put(u,v);
  return retval;
  }

int seqR2::put(long n, vectR2 xya){
  int retval=0;
  if(n>=N){
    fprintf(stderr,"n=%ld larger than seqR1 size N=%ld\n",n,N);
    retval=-1;
    }
  else xy[n]=xya;
  return retval;
  }

/*------------------------------------------------------------------------
 * get a single value from the seqR1 x at location n
 *------------------------------------------------------------------------*/
vectR2 seqR2::get(long n){
  vectR2 xya(0,0);
  if(n<N) xya=xy[n];
  else    fprintf(stderr,"n=%ld larger than seqR1 size N=%ld\n",n,N);
  return xya;
  }

/*------------------------------------------------------------------------
 * get a single value from the seqR1 x,y, or z at location n
 *------------------------------------------------------------------------*/
double seqR2::getx(long n){
  double u=0;
  if(n<N)u=xy[n].getx();
  else    fprintf(stderr,"n=%ld larger than x seqR1 size N=%ld\n",n,N);
  return u;
  }
double seqR2::gety(long n){
  double u=0;
  if(n<N)u=xy[n].gety();
  else    fprintf(stderr,"n=%ld larger than y seqR1 size N=%ld\n",n,N);
  return u;
  }

/*------------------------------------------------------------------------
 * list contents of sequence
 *------------------------------------------------------------------------*/
void seqR2::list(const long start, const long end, const char *str1, const char *str2, FILE *ptr){
  long n,m;
  vectR2 x;
  if(strlen(str1)>0)fprintf(ptr,"%s",str1);
  for(n=start,m=1; n<=end; n++,m++){
    x=get(n);
    fprintf(ptr,"   ");
    x.list(ptr);
    if(m%3==0)fprintf(ptr,"\n");
    }
  if(strlen(str2)>0)fprintf(ptr,"%s",str2);
  }

/*------------------------------------------------------------------------
 * list contents of seqC1 using 1 digit per element
 *------------------------------------------------------------------------*/
void seqR2::list1(void){
  long n,m;
  for(n=0,m=1; n<N; n++,m++){
    printf("(%2.0lf,%2.0lf)    ",getx(n),gety(n));
    if(m%5==0)printf("\n");
    }
  }
```


PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

```
void seqR2::list1(long start, long end){
  long n,m;
  for(n=start,m=1; n<=end; n++,m++){
    printf("(%2.0lf,%2.0lf)    ",getx(n),gety(n));
    if(m%50==0)printf("\n");
    else if(m%10==0)printf(" ");
    }
  }


/*-------------------------------------------------------------------------
 * return the largest pair of values in the seqR1 as measured by norm()
 *-------------------------------------------------------------------------*/
double seqR2::norm(long n){
  vectR2 xya=get(n);
  return xya.norm();
  }

/*-------------------------------------------------------------------------
 * return the largest pair of values in the seqR1 as measured by norm()
 *-------------------------------------------------------------------------*/
vectR2 seqR2::max(int verbose){
  long n;
  double maxnorm=0;
  long   maxn=0;
  vectR2  maxpair;
  for(n=0; n<N; n++) if(norm(n)>maxnorm){maxnorm=norm(n); maxn=n;}
  maxpair=get(maxn);
  if(verbose){
    for(n=0; n<N; n++)
      if(norm(n)>=(maxnorm*0.999))
        printf("max=(%lf,%lf) at n=%ld\n",maxpair.getx(),maxpair.gety(),n);
    }
  return maxpair;
  }

/*=====================================
 * external operations
 *=====================================*/
/*-------------------------------------------------------------------------
 * operator: return p+q
 *-------------------------------------------------------------------------*/
vectR2 operator+(vectR2 p, vectR2 q){
  double px=p.getx();
  double py=p.gety();
  double qx=q.getx();
  double qy=q.gety();
  vectR2 r(px+qx,py+qy);
  return r;
  }

/*-------------------------------------------------------------------------
 * operator: return p-q
 *-------------------------------------------------------------------------*/
vectR2 operator-(vectR2 p, vectR2 q){
  double px=p.getx();
  double py=p.gety();
  double qx=q.getx();
  double qy=q.gety();
  vectR2 r(px-qx,py-qy);
  return r;
  }

/*-------------------------------------------------------------------------
 * operator: return p*q
 *-------------------------------------------------------------------------*/
/*-------------------------------------------------------------------------
 * operator: return -p
 *-------------------------------------------------------------------------*/
vectR2 operator-(vectR2 p){
  double px=p.getx();
  double py=p.gety();
  vectR2  q(-px,-py);
  return q;
  }
```

```
/*------------------------------------------------------------------------
 * operator: return <p> rotated counter-clockwise by <phi> radians
 *------------------------------------------------------------------------*/
vectR2 operator&(vectR2 p,double phi){
  double c,s;
  vectR2 q;
  mat2x2 R; // rotation matrix

  if(phi==0){c=1;         s=0;}
  else      {c=cos(phi); s=sin(phi);}
  R.put(c,-s,s,c);
  q = R*p;
  return q;
  }

/*------------------------------------------------------------------------
 * return the angle theta in radians between the two vectors induced by
 * the points <p> and <q> in the plane R^2
 * on SUCCESS return theta in the closed interval [0:PI]
 * on ERROR   return negative value or exit with value EXIT_FAILURE
 *------------------------------------------------------------------------*/
double pqtheta(const vectR2 p, const vectR2 q){
  const double rp=p.mag(), rq=q.mag();
  double y,theta;
  if(rp==0) return -1;
  if(rq==0) return -2;
  y = (p^q)/(rp*rq);
  if(y>+1.0)   {
    if(y-1<0.0000000000001)y=1.0;
    else{
      fprintf(stderr,"\nERROR using pqtheta(vectR2 p, vectR2 q): (p^q)/(rp*rq)=%.12lf>+1\n",y);
      exit(EXIT_FAILURE);
      }
    }
  if(y<-1.0)
    if(y+1>-0.0000000000001)y=-1.0;
    else{
      fprintf(stderr,"\nERROR using pqtheta(vectR2 p, vectR2 q): (p^q)/(rp*rq)=%.12lf<-1\n",y);
      exit(EXIT_FAILURE);
      }
  theta = acos(y);
  return theta;
  }
```

E.2 Legrange arc distance routines

```
/*==========================================================================
 * Daniel J. Greenhoe
 * header file for routines for Lagrange arcs
 *==========================================================================*/
/*------------------------------------------------------------------------
 * Lagrange arc class
 *------------------------------------------------------------------------*/
class larcc{
  private:
    vectR2 p,q;
  public:
    larcc(vectR2 pp, vectR2 qq){p=pp; q=qq;}          //constructor
    larcc(double px, double py, double qx, double qy){p.put(px,py); q.put(qx,qy);}
    larcc(void){p.put(0,0); q.put(0,0);}
    void   setp(vectR2 pp){p=pp;}
    void   setq(vectR2 qq){q=qq;}
    void   setp(double px, double py){p.put(px,py);}
    void   setq(double qx, double qy){p.put(qx,qy);}
    vectR2  getp(void){return p;}
    vectR2  getq(void){return q;}
    double r(double theta);
    vectR2  x(double theta){return r(theta)*cos(theta);}
    vectR2  y(double theta){return r(theta)*sin(theta);}
    vectR2  xy(double theta);
    double indefint(double theta);// indefinite integral of arc length
    double arclength(void);
    double arclength(long int N);
```


PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

```
    //double  operator|(double x, double y){double z; if(x==0)return 0; else return x/y;}  // division with 0/y = 0 even when y=0
    //double  operator|(double x, double y){double z; if(x==0)z=0; else z=x/y; return z;}  // division with 0/y = 0 even when y=0
  };

/*=====================================
 * prototypes
 *=====================================*/
extern double larc_arclength(double rp, double rq, double tdiff);
extern double larc_indefint(double rp, double rq, double thetap, double thetaq, double theta);
extern double larc_metric(const vectR2 p, const vectR2 q);
extern double larc_metric(const vectR2 p, const vectR2 q, long int N);
extern double larc_metric(const vectR3 p, const vectR3 q);
extern double larc_metric(const vectR4 p, const vectR4 q);
extern double larc_metric(const vectR6 p, const vectR6 q);
//extern vectR2 larc_findq(const vectR2 p, const double theta, const double d, const double minrq, const double maxrq, const double maxerror, const long N);
extern int larc_findq(const vectR2 p, const double theta, const double d, const double minrq, const double maxrq, const double maxerror, const long N, vectR2 *q);
extern vectR3 larc_findq(const vectR3 p, const double theta, const double phi, const double d, const double minrq, const double maxrq, const double maxerror, const long N);
//extern vectR3 larc_findq(vectR3 p, double theta, double phi, double d, long int N);
extern double larc_tau(const double a, const double sigma, const vectR2 p, const vectR2 q, const vectR2 r);
```

```
/*==========================================================================
 * Daniel J. Greenhoe
 * routines for Lagrange arcs
 * Largrange arcs are defined here in a manner analogous to
 * Lagrange polynomial interpolation.
 * Langrange polynomial interpolation is typically defined using
 * Cartesian coordinates in the R^2 plane.
 * Here, "Lagrange arcs" use basically the same idea, but are defined using
 * polar coordinates in the R^2 plane:
 *           y
 *           |   o p        Let (rp,tp) be the polar location of point p.
 *           |  /           where rp is the Euclidean distance from (0,0) to p
 *           | /            and tp is radian measure from the x-axis to p.
 *           |/ tp          Let (rq,tq) be the polar location of point q.
 * ----------+---------- x  The "Lagrange arc" r(theta) is defined here as
 *           |\ tq                     theta -tq         theta -tp
 *           | \            r(theta) = rp ---------- + rq ----------
 *           |  o q                     tp-tq             tq-tp
 *           |
 *==========================================================================*/
/*====================================
 * headers
 *====================================*/
#include<stdio.h>
#include<stdlib.h>
#include<math.h>
#include<main.h>
#include<r1.h>
#include<r2.h>
#include<r3.h>
#include<r4.h>
#include<r6.h>
#include<euclid.h>
#include<larc.h>

/*------------------------------------------------------------------------
 * path length s of Lagrange arc from a point p at polar coordinate (rp,tp)
 *                                 to point q at polar coordinate (rq,tq).
 *          __tq             __tq
 *         |                |              (    dr    )^2
 *    s =  |  ds dtheta =   |   sqrt(r^2 + (--------- )     )  dtheta
 *       __|tp            __|tp            (  dtheta  )
 *
 * reference: Paul Dawkins,
 *   http://tutorial.math.lamar.edu/Classes/CalcII/PolarArcLength.aspx
 *   https://books.google.com/books?id=b4ksCQAAQBAJ&pg=PA533
 *------------------------------------------------------------------------*/
double larc_arclength(double rp, double rq, double tdiff){
  double y;
  const double phi=fabs(tdiff);
```


PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

```
  const double rho=rq-rp;
  const double sp=sqrt(rp*rp*phi*phi+rho*rho);
  const double sq=sqrt(rq*rq*phi*phi+rho*rho);
  const double up=rp*rho*phi+fabs(rho)*sp;
  const double uq= rq*rho*phi+fabs(rho)*sq;
  if(rp==0)     {fprintf(stderr,"\nERROR using larc_arclength(rp,rq,tdiff): rp=%lf\n",      rp);
          exit(EXIT_FAILURE);}
  if(rq==0)     {fprintf(stderr,"\nERROR using larc_arclength(rp,rq,tdiff): rp=%lf\n",      rq);
          exit(EXIT_FAILURE);}
  if(tdiff<=0) {fprintf(stderr,"\nERROR using larc_arclength(rp,rq,tdiff): tdiff=%lf\n",   tdiff);
          exit(EXIT_FAILURE);}
  if(tdiff>PI) {fprintf(stderr,"\nERROR using larc_arclength(rp,rq,tdiff): tdiff=%lf>PI\n",tdiff);
          exit(EXIT_FAILURE);}
  //y = (larc_indefint(rp,rq,0,tdiff,tdiff)-larc_indefint(rp,rq,0,tdiff,0))/tdiff;
  //y2 = (rho>=0)? (fabs(rho)/(2*phi))*(log( rq*phi+sq)-log( rp*phi+sp))
  //             : (fabs(rho)/(2*phi))*(log(-rq*phi+sq)-log(-rp*phi+sp));
  if(fabs(rho)<=0.00000000001) y=rp*phi;
  else{
    if(up<=0){fprintf(stderr,"\nERROR using larc_arclength(rp,rq,tdiff): up=%.20lf rp=%lf rho=%lf
           sp=%lf\n", up,rp,rho,sp);     exit(EXIT_FAILURE);}
    if(uq<=0){fprintf(stderr,"\nERROR using larc_arclength(rp,rq,tdiff): uq=%.20lf rp=%lf rho=%lf
           sp=%lf\n", uq,rq,rho,sq);     exit(EXIT_FAILURE);}
    y = (rq*sq - rp*sp)/(2*rho) + fabs(rho)*(log(uq)-log(up))/(2*phi);
    }
  return y;
  }

/*-------------------------------------------------------------------------
 * indefinite integral for arc length
 * reference: http://integral-table.com/
 *            http://integral-table.com/downloads/integral-table.pdf
 *            indefinite integral (37)
 *            accessed 2015 September 19 12:29PM UTC
 * Note: This function should be viewed as DEPRECATED
 * (that is, don't use it for general computations),
 * However, this function is still useful for testing and verification of
 * larc_metric(vectR2 p, vectR2 q).
 *-------------------------------------------------------------------------*/
double larc_indefint(double rp, double rq, double thetap, double thetaq, double theta){
  double ra = (rp-rq);
  double rb = (rq*thetap-rp*thetaq);
  double a  = ra*ra;
  double b  = 2*ra*rb;
  double c  = ra*ra + rb*rb;
  double x  = theta;
  double y  = (b+2*a*x)/(4*a)*sqrt(a*x*x+b*x+c) +
          (4*a*c-b*b)/(8*a*sqrt(a))*log(2*a*x+b+2*sqrt(a*(a*x*x+b*x+c)));
//double y  = (b+2*a*x)/(4*a)*sqrt(a*x*x+b*x+c) +
          (4*a*c-b*b)/(8*a*sqrt(a))*log(fabs(2*a*x+b+2*sqrt(a*(a*x*x+b*x+c)))); // note: fabs(...)
          is an error in (37)
  return y;
  }

/*-------------------------------------------------------------------------
 * Lagrange arc metric from <p> to <q> in R^2
 *-------------------------------------------------------------------------*/
double larc_metric(const vectR2 p, const vectR2 q){
  const double rp=p.mag(), rq=q.mag();
  const double phi = pqtheta(p,q);
  const vectR2  pq=p-q;
  double d;
  if(rp==0 || rq==0 || phi<=0.0000001){//use Euclidean metric
    d = emetric(p,q);
    //printf("p=(%.2lf,%.2lf) q=(%.3lf,%.3lf) rq=%lf theta=%.12f PI phi=%.12f PI d=%lf
           ae\n",p.getx(),p.gety(),q.getx(),q.gety(),q.mag(),pqtheta(p,q)/PI, phi/PI,d);
    }
  else{//use Lagrange arc length
    d = larc_arclength(rp, rq, phi);
    //printf("p=(%.2lf,%.2lf) q=(%.3lf,%.3lf) rq=%lf theta=%.12f PI phi=%.12f PI d=%lf
           larc\n",p.getx(),p.gety(),q.getx(),q.gety(),q.mag(),pqtheta(p,q)/PI, phi/PI,d);
    }
  return d/PI;
  }

/*-------------------------------------------------------------------------
 * tau function for larc distance function d(p,q)
```


PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

```
 * tau(a,sigma;p,q,r) := 2sigma[ 1/2 d^a(p,r) + 1/2 d^a(r,q) ]^(1/a)
 * reference:
 *   Daniel J. Greenhoe (2016)
 *   "Properties of distance spaces with power triangle inequalities"
 *   Carpathian Mathematical Publications, volume 8, number 1, pages 51--82
 *   doi 10.15330/cmp.8.1.51-82,
 *   http://www.journals.pu.if.ua/index.php/cmp/article/view/483
 *   https://peerj.com/preprints/2055/
 *   https://www.researchgate.net/publication/281831459
 *   section 4: Distance spaces with power triangle inequalities
 *-------------------------------------------------------------------------*/
double larc_tau(const double a, const double sigma, const vectR2 p, const vectR2 q, const vectR2 r){
  double dpr, drq;
  double tau;
  dpr = larc_metric(p,r);
  drq = larc_metric(r,q);
  tau = 2*sigma*pow((0.5*pow(dpr,a) + 0.5*pow(drq,a)),1.0/a);
  return tau;
  }

/*-------------------------------------------------------------------------
 * Lagrange metric from <p> to <q> computed numerically with resolution <N>.
 * Note: This function should be viewed as DEPRECATED
 * (that is, don't use it for general computations),
 * but instead it is strongly recommended to use larc_metric(vectR2 p, vectR2 q).
 * The function larc_metric(vectR2 p, vectR2 q) uses a closed form solution
 * (from an integral lookup table).
 * This function uses a numeric estimation
 * (by an approximated summation along the arc path).
 * However, this function is still useful for testing and verification of
 * larc_metric(vectR2 p, vectR2 q).
 *-------------------------------------------------------------------------*/
double larc_metric(const vectR2 p, const vectR2 q, const long int N){
  larcc arc(p,q);
  double d = arc.arclength(N);
  double ds=d/PI;
  return ds;
  }

/*-------------------------------------------------------------------------
 * Lagrange arc metric from <p> to <q> in R^3
 *-------------------------------------------------------------------------*/
double larc_metric(const vectR3 p, const vectR3 q){
  const double rp=p.mag(), rq=q.mag();
  const double tdiff = pqtheta(p,q);
  const vectR3  pq=p-q;
  double d;
  if(rp==0 || rq==0 || tdiff<=0) d = pq.mag();
  else if(rp==rq)               d = rp*tdiff;
  else                          d = larc_arclength(rp, rq, tdiff);
  return d/PI;
  }

/*-------------------------------------------------------------------------
 * Lagrange arc metric from <p> to <q> in R^3
 *-------------------------------------------------------------------------*/
double larc_metric(const vectR4 p, const vectR4 q){
  const double rp=p.mag(), rq=q.mag();
  const double tdiff = pqtheta(p,q);
  const vectR4  pq=p-q;
  double d;
  if(rp==0 || rq==0 || tdiff<=0) d = pq.mag();
  else if(rp==rq)               d = rp*tdiff;
  else                          d = larc_arclength(rp, rq, tdiff);
  return d/PI;
  }

/*-------------------------------------------------------------------------
 * Lagrange arc metric from <p> to <q> in R^6
 *-------------------------------------------------------------------------*/
double larc_metric(const vectR6 p, const vectR6 q){
  const double rp=p.mag(), rq=q.mag();
  const double tdiff = pqtheta(p,q);
  const vectR6  pq=p-q;
  double d;
  if(rp==0 || rq==0 || tdiff<=0) d = pq.mag();
```

```
  else if(rp==rq)                 d = rp*tdiff;
  else                            d = larc_arclength(rp, rq, tdiff);
  return d/PI;
  }

/*----------------------------------------------------------------------
 * path length of arc computed using numeric integration
 *----------------------------------------------------------------------*/
double larcc::arclength(long int N){
  double sum=0;
  double rp=p.mag(),      rq=q.mag();
  double tdiff=pqtheta(p,q);
  double tp=0, tq=tdiff;
  double theta=tp;
  long int n;
  vectR2 p1,p2;
  double delta=tdiff/(double)N;
  vectR2 pq=p-q;
  double d=pq.mag();//Euclidean distance(p,q)

  if(rp==0)    return d;
  if(rq==0)    return d;
  if(tdiff==0)return d;

  for (n=0; n<N; n++){
    p1 = xy(theta);
    theta += delta;
    p2 = xy(theta);
    sum += chordlength(p1,p2);
    }
  return sum;
  }

/*----------------------------------------------------------------------
 * find the point (x(t),y(t)) on the Lagrange arc larc(p,q) at parameter <theta>
 *----------------------------------------------------------------------*/
vectR2  larcc::xy(double theta){
  double rt=r(theta);
  vectR2 pt(rt*cos(theta),rt*sin(theta));
  return pt;
  }

/*----------------------------------------------------------------------
 * return r(theta) for Lagrange arc(p,q)
 *----------------------------------------------------------------------*/
double  larcc::r(double theta){
  double rp=p.mag();
  double rq=q.mag();
  double tdiff=pqtheta(p,q);
  double tp=0, tq=tdiff;
  double r =rp*(theta-tq)/(tp-tq) + rq*(theta-tp)/(tq-tp); // Lagrange polynomial of theta
  return r;
  }

/*----------------------------------------------------------------------
 * Find a point q in R^2 orientated <phi> with respect to <p>
 * that is within a <maxerror> distance <d> from the point <p>.
 * Search for this point q using <N> search locations
 * over a radial distance from <p> of <minrq> to <maxrq>.
 * If a solution is found, place the point q at <*q> and return 1.
 * If a solution is not found and an apparent discontinuity occurred in
 * in the search, issue a warning and return 0.
 * If a solution is not found and a discontinuity apparently did NOT occur
 * in the search, issue an ERROR message and exit.
 *----------------------------------------------------------------------*/
int larc_findq(const vectR2 p, const double theta, const double d, const double minrq, const double
     maxrq, const double maxerror, const long N, vectR2 *q){
  double rq,dd,ddprev,errord,bestrq,bestd,phi,smallesterror,discon1,discon2;
  vectR2 qq,bestq;
  int discontinuity=0,retval=1;

  qq.polartoxy(minrq,theta);//convert polar coor. to rectangular coordinates
  qq+=p;// search "origin" is the point p (not the R^2 origin (0,0))
  ddprev=larc_metric(p,qq);
  smallesterror=fabs(d-ddprev);

```

```
  for(rq=minrq; rq<=maxrq; rq+=(maxrq-minrq)/(double)N){
    qq.polartoxy(rq,theta);//convert polar coor. to rectangular coordinates
    qq+=p;// search "origin" is the point p (not the R^2 origin (0,0))
    dd=larc_metric(p,qq);
    if(fabs(dd-ddprev)>(maxerror*100)){
      discontinuity=1;
      retval=0;
      discon1=ddprev;
      discon2=dd;
      }
    ddprev=dd;
    errord=fabs(d-dd);
    if(errord<smallesterror){
      bestq=qq;
      bestrq=rq;
      bestd=dd;
      smallesterror=errord;
      phi = pqtheta(p,qq);
      }
    }
  if(smallesterror>maxerror){
    if(discontinuity){
      fprintf(stderr,"\nWARNING using larc_findq(vectR2 p,...): possible discontinuity,\n");
      fprintf(stderr,"  jumping from d=%lf to d=%lf.\n",discon1,discon2);
      fprintf(stderr,"  smallesterror=%lf > %lf=maxerror smallestd=%lf bestrq=%lf theta=%.12lf PI
            phi=%.12lf PI\n",smallesterror,maxerror,bestd,bestrq,theta/PI,phi/PI);
      }
    else{
      fprintf(stderr,"\nERROR using larc_findq(vectR2 p,...): no apparent discontinuity but...\n");
      fprintf(stderr,"  smallesterror=%lf > %lf=maxerror smallestd=%lf bestrq=%lf theta=%.12lf PI
            phi=%.12lf PI\n",smallesterror,maxerror,bestd,bestrq,theta/PI,phi/PI);
      exit(EXIT_FAILURE);
      }
    }
  *q = bestq;
  return retval;
  }

/*-------------------------------------------------------------------------
 * Find the polar length of a point q with radial measure tq that is a
 * distance <d> from the point <p> with polar coordinates (rp,tp)
 * using search resolution <N>
 *-------------------------------------------------------------------------*/
vectR3 larc_findq(const vectR3 p, const double theta, const double phi, const double d, const
        double minrq, const double maxrq, const double maxerror, const long int N){
  double rq,dd,errord,bestrq;
  vectR3 bestq(0,0,0);
  vectR3 q(0,0,0);
  double smallesterror=10000;

  for(rq=minrq; rq<=maxrq; rq+=(maxrq-minrq)/(double)N){
    q.polartoxyz(rq,theta,phi);//convert polar coor. to rectangular coordinates
    q+=p;// search "origin" is the point p (not the R^3 origin (0,0,0))
    dd=larc_metric(p,q);
    errord=fabs(d-dd);
    if(errord<smallesterror){
      bestq=q;
      bestrq=rq;
      smallesterror=errord;
      }
    }
  if(smallesterror>maxerror){
    fprintf(stderr,"\nERROR using larc_findq(vectR3 p,...):\n  smallesterror=%lf > %lf=maxerror
          bestrq=%lf theta=%.2lf PI\n",smallesterror,maxerror,bestrq,theta/PI);
    exit(EXIT_FAILURE);
    }
  return bestq;
  }
```

References

Charalambos D. Aliprantis and Owen Burkinshaw. *Principles of Real Analysis*. Acedemic Press, London, 3 edition, 1998. ISBN 9780120502578. URL http://www.amazon.com/dp/0120502577.

Herbert Amann and Joachim Escher. *Analysis II*. Birkhäauser Verlag AG, Basel–Boston–Berlin, 2008. ISBN 978-3-7643-7472-3. URL http://books.google.com/books?vid=ISBN3764374721.

Tom M. Apostol. *Mathematical Analysis*. Addison-Wesley series in mathematics. Addison-Wesley, Reading, 2 edition, 1975. ISBN 986-154-103-9. URL http://books.google.com/books?vid=ISBN0201002884.

Stefan Banach. Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales (on abstract operations and their applications to the integral equations). *Fundamenta Mathematicae*, 3:133–181, 1922. URL http://matwbn.icm.edu.pl/ksiazki/fm/fm3/fm3120.pdf.

Stefan Banach. *Théorie des opérations linéaires*. Monografje Matematyczne, Warsaw, Poland, 1932a. URL http://matwbn.icm.edu.pl/kstresc.php?tom=1&wyd=10. (Theory of linear operations).

Stefan Banach. *Theory of Linear Operations*, volume 38 of *North-Holland mathematical library*. North-Holland, Amsterdam, 1932b. ISBN 0444701842. URL http://www.amazon.com/dp/0444701842/. English translation of 1932 French edition, published in 1987.

Ladislav Beran. *Orthomodular Lattices: Algebraic Approach*. Mathematics and Its Applications (East European Series). D. Reidel Publishing Company, Dordrecht, 1985. ISBN 90-277-1715-X. URL http://books.google.com/books?vid=ISBN902771715X.

Mihaly Bessenyei and Zsolt Pales. A contraction principle in semimetric spaces. *arXiv.org*, January 8 2014. URL http://arxiv.org/abs/1401.1709.

Leonard Mascot Blumenthal. Distance geometries: a study of the development of abstract metrics. *The University of Missouri studies. A quarterly of research*, 13(2):145, 1938.

Leonard Mascot Blumenthal. *Theory and Applications of Distance Geometry*. Oxford at the Clarendon Press, 1 edition, 1953. ISBN 0-8284-0242-6. URL http://books.google.com/books?vid=ISBN0828402426.

Béla Bollobás. *Linear Analysis; an introductory course*. Cambridge mathematical textbooks. Cambridge University Press, Cambridge, 2 edition, March 1 1999. ISBN 978-0521655774. URL http://books.google.com/books?vid=ISBN0521655773.

Umberto Bottazzini. *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass*. Springer-Verlag, New York, 1986. ISBN 0-387-96302-2. URL http://books.google.com/books?vid=ISBN0387963022.

Bourbaki. *Éléments de mathématique. Prewmieère partie: Les structures fondamentales de l'analyse. Livre I: Théorie des ensembles (Fascicule des résultats)*. Hermann & Cie, Paris, 1939.

Bourbaki. *Theory of Sets*. Elements of Mathematics. Hermann, Paris, 1968. URL http://www.worldcat.org/oclc/281383.

Andrew M. Bruckner, Judith B. Bruckner, and Brian S. Thomson. *Real Analysis*. Prentice-Hall, Upper Saddle River, N.J., 1997. ISBN 9780134588865. URL http://books.google.com/books?vid=ISBN013458886X.

Herbert Busemann. *The Geometry of Geodesics*. Academic Press, 2 edition, 1955. ISBN 0486154629. URL http://books.google.com/books?vid=ISBN0486154629. a Dover 2005 edition has been published which "is an unabridged republication of the work originally published in 1955".

Paul M. Cohn. *Introduction to Ring Theory*. Springer Undergraduate Mathematics Series. Springer Science & Business Media, December 6 2012. ISBN 9781447104759. URL http://books.google.com/books?vid=isbn9781447104759.

Louis Comtet. *Advanced combinatorics: the art of finite and infinite*. D. Reidel Publishing Company, Dordrecht, 1974. ISBN 978-9027704412. URL http://books.google.com/books?vid=ISBN9027704414. translated and corrected version of the 1970 French edition.

Edward Thomas Copson. *Metric Spaces*. Number 57 in Cambridge tracts in mathematics and mathematical physics. Cambridge University Press, London, 1968. ISBN 978-0521047227. URL http://books.google.com/books?vid=ISBN0521047226.

Sheldon W. Davis. *Topology*. McGraw Hill, Boston, 2005. ISBN 007-124339-9. URL http://www.worldcat.org/isbn/0071243399.

Richard Dedekind. Ueber die von drei moduln erzeugte dualgruppe. *Mathematische Annalen*, 53:371–403, January 8 1900. URL http://resolver.sub.uni-goettingen.de/purl/?GDZPPN002257947. Regarding the Dual Group Generated by Three Modules.

Elena Deza and Michel-Marie Deza. *Dictionary of Distances*. Elsevier Science, Amsterdam, 2006. ISBN 0444520872. URL http://books.google.com/books?vid=ISBN0444520872.

Michel-Marie Deza and Elena Deza. *Encyclopedia of Distances*. Springer, Bücher, 3 edition, 2014. ISBN 3662443422. URL http://books.google.com/books?vid=ISBN3662443422.

Jean Alexandre Dieudonné. *Foundations of Modern Analysis*. Academic Press, New York, 1969. ISBN 1406727911. URL http://books.google.com/books?vid=ISBN1406727911.

Tadeusz Dobrowolski and Jerzy Mogilski. Regular retractions onto finite dimensional convex sets. In Krzysztof Jarosz, editor, *Function Spaces: The Second Conference*, volume 172 of *lecture notes in pure and applied mathematics*, pages 85–99, proceedings of the conference at Edwardsville, 1995. Marcel Dekker Inc., New York.

W.D. Duthie. Segments of ordered sets. *Transactions of the American Mathematical Society*, 51(1):1–14, January 1942. . URL http://www.jstor.org/stable/1989978.

Euclid. *Elements*. circa 300BC. URL http://farside.ph.utexas.edu/euclid.html.

L. Euler. De eximio usu methodi interpolationum in serierum doctrina. *Opuscula Analytica Petropoli*, 1: 157–210, 1783. URL http://www.e-rara.ch/zut/content/structure/1197147. English translation of title: "An examination of the use of the method of interpolation in the doctrine of series".

Abraham Fraenkel. Zu den grundlagen der cantor-zermeloschen mengenlehre. *Mathematische Annalen*, 86:230–237, 1922. URL http://dz-srv1.sub.uni-goettingen.de/cache/toc/D36920.html.

Duncan C. Fraser. *Newton's Interpolation Formulas*. C. & E. Layton, London, 1919. URL https://archive.org/details/newtonsinterpola00frasrich.

Maurice René Fréchet. Sur quelques points du calcul fonctionnel (on some points of functional calculation). *Rendiconti del Circolo Matematico di Palermo*, 22:1–74, 1906. Rendiconti del Circolo Matematico di Palermo (Statements of the Mathematical Circle of Palermo).

Maurice René Fréchet. *Les Espaces abstraits et leur théorie considérée comme introduction a l'analyse générale*. Borel series. Gauthier-Villars, Paris, 1928. URL http://books.google.com/books?id=9czoHQAACAAJ. Abstract spaces and their theory regarded as an introduction to general analysis.

Avner Friedman. *Foundations of Modern Analysis*. Holt, Rinehart and Winston, Inc. New York, 1970. ISBN 9780486640624. URL http://books.google.com/books?vid=ISBN0486640620. A 1982 "unabridged and corrected" Dover "republication" has been published.

Fred Galvin and Samuel David Shore. Completeness in semimetric spaces. *Pacific Journal Of Mathematics*, 113(1):67–75, March 1984. . URL http://msp.org/pjm/1984/113-1/pjm-v113-n1-p04-s.pdf.

C. F. Gauss. Theoria interpolationis methodo nova tractata. In *Werke*, volume III, pages 265–327. Königlichen Gesellschaft der Wissenschaften, Göttingen, Germany, 1866.

John Robilliard Giles. *Introduction to the Analysis of Metric Spaces*. Number 3 in Australian Mathematical Society lecture series. Cambridge University Press, Cambridge, 1987. ISBN 0521359287. URL http://books.google.com/books?vid=ISBN0521359287.

John Robilliard Giles. *Introduction to the Analysis of Normed Linear Spaces*. Number 13 in Australian Mathematical Society lecture series. Cambridge University Press, Cambridge, 2000. ISBN 0-521-65375-4. URL http://books.google.com/books?vid=ISBN0521653754.

I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Elsevier, Amsterdam, 7 edition, February 23 2007. ISBN 9780080471112. URL http://books.google.com/books?vid=ISBN0080471110. translated from Russian.

Daniel J. Greenhoe. Order and metric geometry compatible stochastic processing. *PeerJ PrePrints*, February 19 2015. . URL https://peerj.com/preprints/844/. This paper has been submitted to the journal *Stochastic Systems* (http://www.i-journals.org/ssy/) on 2015 February 18 and as of 2016 June 11 (more than 15 months later) no editor decision has yet been received.

Daniel J. Greenhoe. Properties of distance spaces with power triangle inequalities. *Carpathian Mathematical Publications*, 8(1):51–82, 2016a. ISSN 2313-0210. . URL http://www.journals.pu.if.ua/index.php/cmp/article/view/483. preprint versions are available at http://www.researchgate.net/publication/281831459 and https://peerj.com/preprints/2055.

Daniel J. Greenhoe. Order and metric compatible symbolic sequence processing. *PeerJ Preprints*, May 18 2016b. . URL https://peerj.com/preprints/2052v1/.

Norman B. Haaser and Joseph A. Sullivan. *Real Analysis*. Dover Publications, New York, 1991. ISBN 0-486-66509-7. URL http://books.google.com/books?vid=ISBN0486665097.

Paul R. Halmos. *Finite Dimensional Vector Spaces*. Princeton University Press, Princeton, 1 edition, 1948. ISBN 0691090955. URL http://books.google.com/books?vid=isbn0691090955.

Paul Richard Halmos. *Naive Set Theory*. The University Series in Undergraduate Mathematics. D. Van Nostrand Company, Inc., Princeton, New Jersey, 1960. ISBN 0387900926. URL http://books.google.com/books?vid=isbn0387900926.

Felix Hausdorff. *Set Theory*. Chelsea Publishing Company, New York, 3 edition, 1937. ISBN 0828401195. URL http://books.google.com/books?vid=ISBN0828401195. 1957 translation of the 1937 German *Grundzüge der Mengenlehre*.

Robert W. Heath. A regular semi-metric space for which there is no semi-metric under which all spheres are open. *Proceedings of the American Mathematical Society*, 12:810–811, 1961. ISSN 1088-6826. URL http://www.ams.org/journals/proc/1961-012-05/S0002-9939-1961-0125562-9/.

Omar Hijab. *Introduction to Calculus and Classical Analysis*. Undergraduate Texts in Mathematics. Springer, 4 edition, February 9 2016. ISBN 9783319284002. URL http://books.google.com/books?vid=ISBN3319284002.

Alan Jeffrey. *Handbook of Mathematical Formulas and Integrals*. Academic Press, 1 edition, May 19 1995. ISBN 0123825806. URL http://books.google.com/books?vid=ISBN1483295141. "Many of the entries are based on the updated fifth edition of Gradshteyn and Ryzhik's "Tables of Integrals, Series, and Products," though during the preparation of the the book, results were also taken from various other reference works." — Preface, page xix.

Alan Jeffrey and Hui Hui Dai. *Handbook of Mathematical Formulas and Integrals*. Handbook of Mathematical Formulas and Integrals Series. Academic Press, 4 edition, January 18 2008. ISBN 9780080556840. URL http://books.google.com/books?vid=ISBN0080556841.

John Leroy Kelley. *General Topology*. University Series in Higher Mathematics. Van Nostrand, New York, 1955. ISBN 0387901256. URL http://books.google.com/books?vid=ISBN0387901256. Republished by Springer-Verlag, New York, 1975.

Mohamed A. Khamsi and W.A. Kirk. *An Introduction to Metric Spaces and Fixed Point Theory*. John Wiley, New York, 2001. ISBN 978-0471418252. URL http://books.google.com/books?vid=isbn0471418250.

A. Korselt. Bemerkung zur algebra der logik. *Mathematische Annalen*, 44(1):156–157, March 1894. ISSN 0025-5831. . URL http://www.springerlink.com/content/v681m56871273j73/. referenced by Birkhoff(1948)p.133.

Carlos S. Kubrusly. *The Elements of Operator Theory*. Springer, 1 edition, 2001. ISBN 9780817641740. URL http://books.google.com/books?vid=ISBN0817641742.

Kazimierz Kuratowski. Sur la notion d'ordre dans la theorie des ensembles (on the concept of order in set theory). *Fundamenta Mathematicae*, 2:161–171, 1921. URL http://matwbn.icm.edu.pl/ksiazki/fm/fm2/fm2122.pdf.

Kazimierz Kuratowski. *Introduction to Set Theory and Topology*, volume 13 of *International Series on Monographs on Pure and Applied Mathematics*. Pergamon Press, New York, 1961. URL http://www.worldcat.org/oclc/1023273.

J. L. Lagrange. Leçons élémentaires sur les mathématiques données à l'école normale en 1795. In *Œuvres complètes*, volume 7, pages 187–288. Journal de l'École polytechnique, 1877. URL http://sites.mathdoc.fr/cgi-bin/oeitem?id=OE_LAGRANGE__7_183_0.

Nicolas K. Laos. *Topics in Mathematical Analysis and Differential Geometry*, volume 24 of *Series in pure mathematics*. World Scientific, 1998. ISBN 9789810231804. URL http://books.google.com/books?vid=ISBN9810231806.

M. Loève. *Probability Theory I*, volume 45 of *Graduate Texts in Mathematics*. 4 edition, 1977. ISBN 9780387902104. URL http://books.google.com/books?vid=ISBN0387902104.

Saunders MacLane and Garrett Birkhoff. *Algebra*. AMS Chelsea Publishing, Providence, 3 edition, 1999. ISBN 0821816462. URL http://books.google.com/books?vid=isbn0821816462.

I. J. Maddox. *Elements of Functional Analysis*. Cambridge University Press, Cambridge, 2, revised edition, 1989. ISBN 9780521358682. URL books.google.com/books?vid=ISBN052135868X.

Roger Duncan Maddux. *Relation Algebras*. Elsevier Science, 1 edition, July 27 2006. ISBN 0444520139. URL http://books.google.com/books?vid=ISBN0444520139.

John H. Matthews and Kurtis D. Fink. *Numerical Methods Using MatLab 3rd edition*. Prentice-Hall, Inc., Upper River, NJ, 1992. ISBN 0-13-013164-4.

Erik Meijering. A chronology of interpolation: From ancient astronomy to modern signal and image processing. *Proceedings of the IEEE*, 90(3):319–342, March 2002. URL http://bigwww.epfl.ch/publications/meijering0201.pdf.

Karl Menger. Untersuchungen über allgemeine metrik. *Mathematische Annalen*, 100:75–163, 1928. ISSN 0025-5831. URL http://link.springer.com/article/10.1007/BF01455705. (Investigations on general metric).

Anthony N. Michel and Charles J. Herget. *Applied Algebra and Functional Analysis*. Dover Publications, Inc., 1993. ISBN 0-486-67598-X. URL http://books.google.com/books?vid=ISBN048667598X. original version published by Prentice-Hall in 1981.

Dragoslav S. Mitrinović, J. E. Pečarić, and Arlington M. Fink. *Classical and New Inequalities in Analysis*, volume 61 of *Mathematics and its Applications (East European Series)*. Kluwer Academic Publishers, Dordrecht, Boston, London, 2010. ISBN 978-90-481-4225-5. URL http://www.amazon.com/dp/0792320646.

Ilya S. Molchanov. *Theory of Random Sets*. Probability and Its Applications. Springer, 2005. ISBN 185233892X. URL http://books.google.com/books?vid=ISBN185233892X.

James R. Munkres. *Topology*. Prentice Hall, Upper Saddle River, NJ, 2 edition, 2000. ISBN 0131816292. URL http://www.amazon.com/dp/0131816292.

NASA. Reference guide to the international space station, utilization edition. Technical report, National Aeronautics and Space Administration, September 2015. URL http://www.nasa.gov/sites/default/files/atoms/files/np-2015-05-022-jsc-iss-guide-2015-update-111015-508c.pdf.

Isaac Newton. Methodus differentialis. In *Analysis Per Quantitatum Series, Fluxiones ac Differentias : cum Enumeratio Linearum Tertii Ordinis*. 1711. "A photographic reproduction of the original Latin text …and a translation are given" in Fraser (1919), pages 9–17 (Latin) and pages 18–25 (English).

Timothy S. Norfolk. When does a metric generate convex balls? not sure about the year, 1991. URL http://www.math.uakron.edu/~norfolk/.

Timur Oikhberg and Haskell Rosenthal. A metric characterization of normed linear spaces. *Rocky Mountain Journal Of Mathematics*, 37(2):597–608, 2007. URL http://www.ma.utexas.edu/users/rosenthl/pdf-papers/95-oikh.pdf.

Oystein Ore. On the foundation of abstract algebra. i. *The Annals of Mathematics*, 36(2):406–437, April 1935. URL http://www.jstor.org/stable/1968580.

Giuseppe Peano. *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazioni della logica deduttiva.* Fratelli Bocca Editori, Torino, 1888a. Geometric Calculus: According to the *Ausdehnungslehre* of H. Grassmann.

Giuseppe Peano. *Geometric Calculus: According to the Ausdehnungslehre of H. Grassmann.* Springer (2000), 1888b. ISBN 0817641262. URL http://books.google.com/books?vid=isbn0817641262. originally published in 1888 in Italian.

John Ratcliffe. *Foundations of Hyperbolic Manifolds*, volume 149 of *Graduate Texts in Mathematics.* Springer Science & Business Media, 2013. ISBN 9781475740134. URL http://books.google.com/books?vid=ISBN1475740131.

Stefan Rolewicz. *Metric Linear Spaces.* Mathematics and Its Applications. D. Reidel Publishing Company, 1 edition, 1985. ISBN 9789027714800. URL http://books.google.com/books?vid=ISBN9027714800.

Alex Ruzicka, Jeffrey Grossman, Audrey Bouvier, Christopher D.K. Herd, and Carl B. Agee. The meteoritical bulletin. Technical Report 102, The Meteoritical Society, August 19 2015. URL http://www.lpi.usra.edu/meteor/docs/mb102.pdf.

Eric Schechter. *Handbook of Analysis and Its Foundations.* Academic Press, 1996. ISBN 9780080532998. URL http://books.google.com/books?vid=ISBN0080532993.

Judith Silver and Erik Stokes. A great circle metric. In *Proceedings of the West Virginia Academy of Science*, volume 78, pages 8–13, 2007. URL http://www.marshall.edu/wvas/public.html.

James Stewart. *Single Variable Essential Calculus.* Cenage Learning, 2 edition, February 10 2012. ISBN 9781133112761. URL http://books.google.com/books?vid=ISBN1133112765.

Patrick Suppes. *Axiomatic Set Theory.* Dover Publications, New York, 1972. ISBN 0486616304. URL http://books.google.com/books?vid=ISBN0486616304.

Wolfgang J. Thron. *Topological structures.* Holt, Rinehart and Winston, New York, 1966. URL http://books.google.com/books?id=JRM_AAAAIAAJ.

M.L.J. van de Vel. *Theory of Convex Structures*, volume 50 of *North-Holland Mathematical Library.* North-Holland, Amsterdam, 1993. ISBN 978-0444815057. URL http://books.google.com/books?vid=ISBN0444815058.

Edward Waring. Problems concerning interpolations. *Philosophical Transactions of the Royal Society of London*, 69:59–67, 1779. URL http://www.jstor.org/stable/106408.

Norbert Wiener. A simplification of the logic of relations. *Proceedings of the Cambridge Philosophical Society*, 17:387–390, 1914. URL http://books.google.com/books?vid=ISBN0674324498&pg=PA224. reprinted in Jean Van Heijenoort's *From Frege to G odel : A Source Book in Mathematical Logic, 1879-1931* page 224–227.

Wallace Alvin Wilson. On semi-metric spaces. *American Journal of Mathematics*, 53(2):361–373, April 1931. URL http://www.jstor.org/stable/2370790.

Robert S. Wolf. *Proof, Logic, and Conjecture; The Mathematician's Toolbox.* W.H. Freeman and Company, 1998. ISBN 0716730502. URL http://www.worldcat.org/isbn/0716730502.

Don Yeomans and Paul Chodas. Additional details on the large fireball event over russia on feb. 15, 2013. Technical report, Near Earth Object Program, March 05 2013. URL http://neo.jpl.nasa.gov/news/fireball_130301.html.

Ernst Zermelo. Untersuchungen über die grundlagen der mengenlehre i. *Mathematische Annalen*, 65: 261–281, 1908a. URL http://dz-srv1.sub.uni-goettingen.de/sub/digbib/loader?ht=VIEW&did=D38200.

Ernst Zermelo. Investigations in the foundations of set theory. In Jean Van Heijenoort, editor, *From Frege to G odel : A Source Book in Mathematical Logic, 1879-1931*, pages 199–215. Harvard University Press, 1967, Cambridge, Massechusettes, 1908b. URL http://www.amazon.com/dp/0674324498.

Reference Index

Subject Index

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

PeerJ Preprints | https://doi.org/10.7287/peerj.preprints.2467v1 | CC BY 4.0 Open Access | rec: 22 Sep 2016, publ: 22 Sep 2016

dgreenhoe@gmail.com

Received: aa bb 20YY Revised: cc dd 20ZZ