The iterated Data Rate Theorem for unstable biological dynamics

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Abstract

Counterintuitively, unstable control systems can allow extremely rapid responses that may be strongly selected by evolutionary process. Application of an information bottleneck iteration to the Data Rate Theorem – taking the minimum necessary rate of control information as a distortion measure – leads to a diffusion dynamic for onset of sudden failure in such systems via the necessary convexity of the Rate Distortion Function. Imposition of maintenance mechanisms seems a necessary consequence, but those too are subject to deterioration by aging or pathological exposures. In sum, a fairly simple control theory model that iterates the Data Rate Theorem provides deep insight across a wide sweep of diseases and the chronic dysfunctions of senescence.

Key Words: aging, chronic disease, Data Rate Theorem, information bottleneck, Rate Distortion Theorem, regulatory failure

1 Introduction

Multicellularity itself is inherently unstable: cancerous 'cheating' is expected to be a persistent and characteristic bane of multi-celled organisms (e.g., Aktipis et al. 2015). Nunney (1999) has explored cancer occurrence as a function of animal size, suggesting that in larger animals, whose lifespan grows as about the 4/10 power of their cell count, prevention of cancer in rapidly proliferating tissues becomes more difficult in proportion to size. Cancer control requires the development of additional mechanisms and systems to address tumorigenesis as body size increases – a synergistic effect of cell number and organism longevity. This pattern may represent a real barrier to the evolution of large, long-lived animals, and Nunney predicts that those that do evolve have recruited additional controls over those of smaller animals to prevent cancer.

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Different tissues may have evolved markedly different tumor control strategies. All such are likely to be energetically expensive, permeated with different complex signaling strategies, and subject to a multiplicity of reactions to signals, including, in social animals, those related to psychosocial stress.

The immune system, a sophisticated subcomponent of the more general tumor control system (Atlan and Cohen 1998), is also inherently unstable: failure of differentiation between 'self' and 'nonself' leads to carcinogenic chronic inflammation (e.g., Rakoff-Nahoum 2006) and autoimmune disorders (e.g., Mackey and Rose 2014). The immune system must, then, both respond quickly to challenge and be closely regulated to avoid such catastrophes.

Rau and Elbert (2001) review blood pressure regulation, particularly the interaction between baroreceptor activation and central nervous function. Unregulated blood pressure, of course, would be quickly fatal in any animal with a circulatory system, a matter as physiologically fundamental as tumor control and control of the immune response. The reflex is not simple: it may be inhibited through peripheral processes, for example under conditions of high metabolic demand. Higher brain structures modulate the reflex, for instance, when threat is detected and fight or flight responses are being prepared. This suggests, then, that blood pressure control is a broad and actively regulated modular physiological system.

Emotions, following Thayer and Lane (2000), are an integrative index of individual adjustment to changing environmental demands, an organismal response to an environmental event that allows rapid mobilization of multiple subsystems. Emotions allow the efficient coordination of the organism for goal-directed behavior. When the system works properly, it allows for flexible adaptation of the organism to changing environmental demands. An emotional response must be regulated to represent a proper selection of an appropriate response and the inhibition of other less appropriate responses from a more or less broad behavioral repertoire of possible responses.

According to Damasio (1998), emotion is the most complex expression of homeostatic regulatory systems.

From the perspective of Thayer and Lane (2000), disorders of affect represent a condition in which the individual is unable to select the appropriate response, or to inhibit the inappropriate response, so that the response selection mechanism is somehow corrupted – regulation fails.

Gilbert (2001) similarly suggests that a canonical form of such corruption is the excitation of modes that, in other circumstances, represent normal evolutionary adaptations, representing a fundamental failure of regulation.

Some thought will identify a number of other rapidly acting but inherently unstable dynamic systems whose failure of regulation induces pathology, for example the hypothalamic-pituitary-adrenal (HPA) axis in the etiology of post-traumatic stress disorder (PTSD). Indeed, 'code' dynamics of basic gene expression, protein folding, and communication at the glycan/lectin interface – many of which must take place relatively rapidly – may also be inherently unstable, reqiring draconian levels of regulation and control (Wallace 2015a).

Inherently unstable systems are not anomalies, as failed atomistic Western

economic theory would seem to imply (Wallace 2015b, Ch. 1). Inherent instability allows rapid responses that have been strongly selected for. See Ristroph et al. (2013) for examples, a more complete discussion, and references.

How do we understand the regulation of inherently unstable control systems? Two fundamental relations, the Data Rate and Rate Distortion Theorems, are a necessary foundation. Their convolution, we shall show, provides deeper insight than either alone.

2 The Data Rate Theorem

The Data Rate Theorem (DRT), a generalization of the classic Bode Integral Theorem for linear control systems, describes the stability of feedback control under data rate constraints (Nair et al. 2007). Given a noise-free data link between a discrete linear plant and its controller, unstable modes can be stabilized only if the feedback data rate $\mathcal{H}$ is greater than the rate of 'topological information' generated by the unstable system. For the simplest incarnation, if the linear matrix equation of the plant is of the form $x_{t+1} = \mathbf{A}x_t + ...$, where x_t is the n-dimensional state vector at time t, then the necessary condition for stabilizability is that

$$\mathcal{H} > \log[|det\mathbf{A}^u|] \tag{1}$$

where det is the determinant and $\mathbf{A}^u$ is the decoupled unstable component of $\mathbf{A}$, i.e., the part having eigenvalues ≥ 1. The determinant represents a generalized volume. Thus there is a critical positive data rate below which there does not exist any quantization and control scheme able to stabilize an unstable system (Nair et al. 2007).

The new theorem, and its variations, relate control theory to information theory and are as fundamental as the Shannon Coding and Source Coding Theorems, and the Rate Distortion Theorem for understanding complex cognitive machines and biological phenomena (Cover and Thomas 2006).

3 The Rate Distortion Theorem

Suppose a sequence of signals is generated by a biological information source Y having output $y^n = y_1, y_2, ...$ This is 'digitized' in terms of the observed behavior of the system with which it communicates, for example a sequence of 'observed behaviors' $b^n = b_1, b_2, ...$ Assume each b^n is then deterministically retranslated back into a reproduction of the original biological signal, $b^n \rightarrow \hat{y}^n = \hat{y}_1, \hat{y}_2, ...$

Define a distortion measure $d(y, \hat{y})$ comparing the original to the retranslated path. Many distortion measures are possible. For example, the Hamming distortion is defined simply as $d(y, \hat{y}) = 1, y \neq \hat{y}, d(y, \hat{y}) = 0, y = \hat{y}$.

For continuous variates, the squared error distortion measure is just $d(y, \hat{y}) = (y - \hat{y})^2$.

There are, obviously, many possible distortion measures. The distortion between paths y^n and $\hat{y}^n$ is defined as

$$d(y^n, \hat{y}^n) \equiv \frac{1}{n} \sum_{j=1}^{n} d(y_j, \hat{y}_j) \tag{2}$$

A remarkable characteristic of the Rate Distortion Theorem is that the basic result is independent of the exact distortion measure chosen (Cover and Thomas 2006). Indeed, below we shall iterate the Data Rate Theorem via the information bottleneck method of Tishby et al. (1999), and use $\mathcal{H}$ of the Data Rate Theorem as our distortion measure.

Suppose that with each path y^n and b^n-path retranslation into the y-language, denoted $\hat{y}^n$, there are associated individual, joint, and conditional probability distributions $p(y^n), p(\hat{y}^n), p(y^n, \hat{y}^n), p(y^n|\hat{y}^n)$.

The average distortion is defined as

$$D \equiv \sum_{y^n} p(y^n) d(y^n, \hat{y}^n) \tag{3}$$

It is possible to define the information transmitted from the Y to the $\hat{Y}$ process using the Shannon source uncertainty of the strings:

$$I(Y, \hat{Y}) \equiv H(Y) - H(Y|\hat{Y}) = H(Y) + H(\hat{Y}) - H(Y, \hat{Y}) \tag{4}$$

where $H(..., ...)$ is the standard joint, and $H(...|...)$ the conditional, Shannon uncertainties (Cover and Thomas 2006).

If there is no uncertainty in Y given the retranslation $\hat{Y}$, then no information is lost, and the systems are in perfect synchrony.

In general, of course, this will not be true.

The rate distortion function $R(D)$ for a source Y with a distortion measure $d(y, \hat{y})$ is defined as

$$R(D) = \min_{p(y,\hat{y}); \sum_{(y,\hat{y})} p(y)p(y|\hat{y})d(y,\hat{y}) \leq D} I(Y, \hat{Y}) \tag{5}$$

The minimization is over all conditional distributions $p(y|\hat{y})$ for which the joint distribution $p(y, \hat{y}) = p(y)p(y|\hat{y})$ satisfies the average distortion constraint (i.e., average distortion $\leq D$).

The Rate Distortion Theorem states that $R(D)$ is the minimum necessary rate of information transmission which ensures the communication between the biological vesicles does not exceed average distortion D. Thus $R(D)$ defines a minimum necessary channel capacity. Cover and Thomas (2006) or Dembo and Zeitouni (1998) provide details. The rate distortion function has been calculated for a number of systems, often using Lagrange multiplier or Khun-Tucker optimization methods.

Cover and Thomas (2006, Lemma 13.4.1) show that $R(D)$ is necessarily a decreasing convex function of D for any reasonable definition of distortion.

That is, $R(D)$ is always a reverse J-shaped curve. This will prove crucial for the overall argument: convexity is an exceedingly powerful mathematical condition, and permits deep mathematical inference (e.g., Rockafellar 1970). Ellis (1985, Ch. VI) applies convexity theory to conventional statistical mechanics. This is, indeed, the central point from which all else in this paper will follows. We will use the Gaussian channel as an easily calculated example, but the central results are quite general, and will drive the final argument.

For the standard Gaussian channel having noise with zero mean and variance σ^2, using the squared distortion measure,

$$R(D) = 1/2 \log[\sigma^2/D], 0 \leq D \leq \sigma^2$$
$$R(D) = 0, D > \sigma^2 \tag{6}$$

4 Simple Rate Distortion dynamics

Following Wallace (2015a), for the Gaussian channel, we define a 'Rate Distortion entropy' as the Legendre transform

$$S_R = R(D) - DdR(D)/dD = 1/2 \log[\sigma^2/D] + 1/2 \tag{7}$$

The simplest nonequilibrium Onsager equation (de Groot and Mazur 1984) is then

$$dD/dt = -\mu dS_R/dD = \mu/2D \tag{8}$$

where t is the time and μ is a diffusion coefficient. By inspection,

$$D(t) = \sqrt{\mu t} \tag{9}$$

which is the classic solution to the diffusion equation. Such 'correspondence reduction' serves as a basis to argue upward in both scale and complexity.

But regulation does not involve the diffusive drift of average distortion. Let M be the rate of metabolic free energy available for simple regulation. Then a plausible model, in the presence of an internal system noise β^2 in addition to the environmental channel noise defined by σ^2, is the stochastic differential equation

$$dD_t = (\frac{\mu}{2D_t} - F(M))dt + \frac{\beta^2}{2} D_t dW_t \tag{10}$$

where dW_t represents unstructured white noise and $F(M) \geq 0$ is monotonically increasing in M.

This has the nonequilibrium steady state expectation

$$D_{nss} = \frac{\mu}{2F(M)} \tag{11}$$

Using the Ito chain rule on equation (10) (Protter 1990; Khasminskii 2012), one can calculate the variance in the distortion as $E(D_t^2) - (E(D_t))^2$.

Letting $Y_t = D_t^2$ and applying the Ito relation,

$$dY_t = [2\sqrt{Y_t}(\frac{\mu}{2\sqrt{Y_t}} - F(M) + \frac{\beta^4}{4}Y_t]dt + \beta^2 Y_t dW_t \tag{12}$$

where $(\beta^4/4)Y_t$ is the Ito correction to the time term of the SDE.

No real number solution for the expectation of $Y_t = D_t^2$ can exist unless the discriminant of the resulting quadratic equation is ≥ 0, producing a minimum necessary rate of available metabolic free energy for regulatory stability defined by

$$F(M) \geq \frac{\beta^2}{2}\sqrt{\mu} \tag{13}$$

Values of $F(M)$ below this limit will trigger a phase transition into a disintegrated, pathological, system dynamic in a highly punctuated manner. Wallace (2015a) uses a Black-Scholes model to calculate the form of $F(M)$, and to solve for M in terms of system parameters. Applying the inverse of the function F to equation (13) gives a slightly different form of the Data Rate Theorem, in terms of minimum necessary metabolic free energy.

The so-called 'natural' channel, having the Rate Distortion Function $R(D) = \beta/D$, gives the relations

$$D(t) = (6\beta\mu t)^{1/3} \tag{14}$$

and

$$D_{nss} = \sqrt{\frac{2\beta\mu}{F(M)}} \tag{15}$$

We can extend the Data Rate Theorem by iterating these arguments for any convex Rate Distortion Function, via the information bottleneck method.

5 The iterated Data Rate Theorem

The Data Rate Theorem states that there is a minimum necessary rate of control information needed to stabilize an inherently unstable system. Is this, in itself, a stable condition? That is, once a stabilizing control information rate has been identified, is that the whole story? A control system has at least three components: the structure to be controlled, the mechanism for control, and the underlying 'program' of that mechanism. The 'mechanism', in our case, is an interacting set of physiological systems. With time, the structure and mechanism may change – evolve, degrade, age, and so on, requiring increased rates of control information for continued proper function. This is a fundamental – and familiar – problem in system maintenance.

An approach to the dynamics of control stability is possible using a variant of the information bottleneck of Tishby et al. (1999).

We envision an iterated application of the Rate Distortion Theorem to a control system in which a series of 'orders' $y^n = y_1, ..., y_n$, having probability $p(y^n)$, is sent through and the outcomes monitored as $\hat{y}^n = \hat{y}_1, ..., \hat{y}_n$. The

distortion measure, however, is now taken as the minimum necessary control information $\mathcal{H}(y^n, \hat{y}^n)$, defining an average 'distortion' $\hat{\mathcal{H}}$ as

$$\hat{\mathcal{H}} = \sum_{y^n} p(y^n)\mathcal{H}(y^n, \hat{y}^n) \geq 0 \tag{16}$$

We can then define a new, iterated, Rate Distortion Function $\mathcal{R}(\hat{\mathcal{H}})$ and a new 'entropy' as

$$\mathcal{S} = \mathcal{R}(\hat{\mathcal{H}}) - \hat{\mathcal{H}} d\mathcal{R}/d\hat{\mathcal{H}} \tag{17}$$

We next invoke the analog to the 'diffusion' equation (8),

$$d\hat{\mathcal{H}}/dt = -\mu d\mathcal{S}/d\hat{\mathcal{H}} \tag{18}$$

where t is the time and μ a diffusion coefficient.

Since $\mathcal{R}$ is always a convex function of $\hat{\mathcal{H}}$ (Tishby et al. 1999; Cover and Thomas 2006), this relation has the solution

$$\hat{\mathcal{H}}(t) = f(t) \tag{19}$$

where $f(t)$ is a monotonic increasing function of t. Thus, in the absence of continuous maintenance, the needed control signal will relentlessly rise in time, surpassing all possible bounds, and hence triggering a failure to control an inherently unstable system.

The next stage of the argument involves generalization of equation (18) in the direction of equation (10) to allow calculation of a nonequilibrium steady state in a model for ongoing, continuous investment metabolic energy and other resources at a rate M in order to maintain the system. As a further consequence of the convexity of the Rate Distortion Function, this too will have an expectation analogous to equations (11) and (15), so that

$$\hat{\mathcal{H}}_{nss} \propto 1/g(M) \tag{20}$$

where $g(M)$ is monotonic increasing in the metabolic energy rate M.

Since we do not know the mathematical form of $\mathcal{R}(\hat{\mathcal{H}})$, we cannot carry out the calculation leading to equation (13) or the explicit results of Wallace (2015a), based on the assumption of a Gaussian channel.

6 Discussion and conclusions

Control system instability allows characteristically rapid responses that have been strongly selected by evolutionary process at various scales and levels of organization. Our application of an information bottleneck iteration to the Data Rate Theorem – taking the minimum necessary rate of control information $\mathcal{H}$ as a distortion measure – leads to the diffusion dynamic of equation (19), via the necessary convexity of the Rate Distortion Function. This suggests that a spectrum of critical but inherently unstable physiological and psychosocial

mechanisms in higher animals is subject to natural or induced deterioration leading to the inevitable breakdown of regulation by aging or disease. Imposition of maintenance constraints on such deterioration, via equation (20) seems a necessary consequence, but that too is subject to deterioration by aging or pathological exposures, or synergisms between them.

In sum, a fairly simple control theory model that iterates the Data Rate Theorem appears to provide deep insight across a wide sweep of diseases and the chronic dysfunctions of senescence. While earlier studies in this series broadly examined the cognitive nature of codes and control at varous scales and levels of organization via the dual information source associated with such processes (e.g., Wallace 2015a, b; Wallace and Wallace 2010), the narrow focus here is on the properties of inherently unstable control structures, a small but important subset of cognitive systems.

There are more general implications to the approach. For example, one possible inference is that consciousness in higher animals, a very rapid cognitive process that typically acts with a 100 ms time constant, is inherently unstable and must be closely regulated (Wallace 2005). On larger scales, Wallace and Fullilove (2015) use similar methodology to examine the institutional dynamics of American Apartheid, adapting the theory of the firm from evolutionary economics (Wallace 2015b), and forthcoming work suggests applications in real-time critical system automation (Wallace 2016).

Engineering equilibria in natural systems – return to a stable mode after perturbation – may be rare indeed, given the hyperresponsiveness of unstable control mechanisms. As is increasingly well understood, equilibrium formulations of economic process fail to reflect real-world patterns (Wallace 2015b). Unstable control systems may be favored by evolutionary selection pressures for a considerable array of natural – and sociocultural – phenomena, greatly raising the importance of embedding regulatory mechanisms – 'junk DNA' and its analogs – across scales and levels of organization.

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