Estimating the average daily rainfall in Thailand using confidence intervals for the common mean of several delta-lognormal distributions

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ABSTRACT

The daily average natural rainfall amounts in the five regions of Thailand can be estimated using the confidence intervals for the common mean of several delta-lognormal distributions based on the fiducial generalized confidence interval (FGCI), large sample (LS), method of variance estimates recovery (MOVER), parametric bootstrap (PB), and highest posterior density intervals based on Jeffreys' rule (HPD-JR) and normal-gamma-beta (HPD-NGB) priors. Monte Carlo simulation was conducted to assess the performance in terms of the coverage probability and average length of the proposed methods. The numerical results indicate that MOVER and PB provided better performances than the other methods in a variety of situations, even when the sample case was large. The efficacies of the proposed methods were illustrated by applying them to real rainfall datasets from the five regions of Thailand.

INTRODUCTION

Approximately 82.2% of Thailand's cultivated land area depends on natural rainfall (Supasod, 2006), thereby indicating its importance for Thai agriculture. However, it is a natural phenomenon with a significant level of uncertainty that can cause natural disasters such as droughts, floods, and landslides. In many countries around the world, extreme rainfall events have been increasing in frequency and duration. On December 5, 2017, Storm Desmond led to heavy rainfall causing flooding in northern England, Southern Scotland, and Ireland (Otto and Oldenborgh, 2017). On July 6–7, 2018, extreme rainfall events such as floods and landslides affected over 5,000 houses, and approximately 1.9 million people in Japan were evacuated from the at-risk area (Oldenborgh, 2018). In mid-September 2019, the amount of rainfall was extreme during Tropical Storm Imelda in Southeast Texas, USA, where over 1,000 people were affected by large-scale flooding and there were 5 deaths (Oldenborgh et al., 2019). Thus, it is necessary to assess how rainfall varies in each region of a country on a daily basis. Due to the climate pattern and meteorological conditions, Thailand is commonly separated into five regions: northern, northeastern, central, eastern, and southern. The rainfall in each region varies widely due to both location and seasonality. Importantly, Thailand's rainfall data include many zeros with probability $\delta > 0$ and positive right-skewed data following a lognormal distribution for the remainder of the probability. Thus, applying a delta-lognormal distribution (Aitchison, 1955) is appropriate.

The mean is a measure of the center of a set of observations (Casella and Berger, 2002) that can be used in statistical inference, while functions of the mean such as the ratio or difference between two means can also be used. These parameters have been applied in many research areas, such as medicine, fish stocks, pharmaceutics, and climatology. For example, they have been used for hypothesis testing of the effect of race on the average medical costs between African American and Caucasian patients with type I diabetes (Zhou et al., 1997), to estimate the mean charges for diagnostic tests on patients with

⁴⁵ unstable chronic medical conditions (Zhou and Tu, 2000; Tian, 2005; Tian and Wu, 2007; Li et al., 2013),
⁴⁶ to estimate the maximum alcohol concentration in men in an alcohol interaction study (Tian and Wu,
⁴⁷ 2007; Krishnamoorthy and Oral, 2015), to estimate the mean red cod density around New Zealand as
⁴⁸ an indication of fish abundance (Fletcher, 2008; Wu and Hsieh, 2014), and to estimate the mean of the
⁴⁹ monthly rainfall totals to compare rainfall in Bloemfontein and Kimberley in South African (Harvey and
⁵⁰ van der Merwe, 2012).

⁵¹ In practice, the mean has been widely used in many fields, as mentioned before. When independent
⁵² samples are recorded from several situations, then the common mean is of interest when studying more
⁵³ than one population. Many researchers have investigated methods for constructing confidence interval
⁵⁴ (CIs) for the common mean of several distributions. For example, Fairweather (1972) proposed a linear
⁵⁵ combination of Student's t to construct CIs for the common mean of several normal distributions. Jordan
⁵⁶ and Krishnamoorthy (1996) solved the problem of CIs for the common mean under unknown and
⁵⁷ unequal variances based on Student's t and independent F variables from several normal populations.
⁵⁸ Krishnamoorthy and Mathew (2003) presented the generalized CI (GCI) and compared it with the CIs
⁵⁹ constructed by Fairweather (1972), and Jordan and Krishnamoorthy (1996). Later, Lin and Lee (2005)
⁶⁰ developed a GCI for the common mean of several normal populations. Tian and Wu (2007) provided CIs
⁶¹ for the common mean of several lognormal populations using the generalized variable approach, which
⁶² was shown to be consistently better than the large sample (LS) approach. Lin and Wang (2013) studied
⁶³ the modification of the quadratic method to make inference via hypothesis testing and interval estimation
⁶⁴ for several lognormal means. Krishnamoorthy and Oral (2015) proposed the method of variance estimates
⁶⁵ recovery (MOVER) approach for the common mean of lognormal distributions.

⁶⁶ As mentioned earlier, many researchers have developed CIs for the common mean of several normal
⁶⁷ and lognormal distributions. However, there has not yet been an investigation of statistical inference
⁶⁸ using the common mean of several delta-lognormal distributions. Since the common mean is used to
⁶⁹ study more than one population, the average precipitation in the five regions in Thailand can be estimated
⁷⁰ using it as there is an important need to estimate the daily rainfall trends in these regions. Furthermore,
⁷¹ the daily rainfall records from the five regions in Thailand satisfy the assumptions for a delta-lognormal
⁷² distribution. Herein, CIs for the common mean of several delta-lognormal models based on the fiducial
⁷³ GCI (FGCI), LS, MOVER, parametric bootstrap (PB), and highest posterior density (HPD) intervals
⁷⁴ based on Jeffreys' rule (HPD-JR) and normal-gamma-beta (HPD-NGB) priors are proposed. The outline
⁷⁵ of this article is as follows. The ideas behind the proposed methods are detailed in Section 2. Numerical
⁷⁶ computations are reported in Section 3. In Section 4, the daily natural rainfall records of the five regions
⁷⁷ in Thailand are used to illustrate the efficacy of the methods. Finally, the paper is ended with a discussion
⁷⁸ and conclusions.

METHODS

⁸⁰ Let $W_{ij} = (W_{i1}, W_{i2}, ..., W_{in_i})$ be random samples drawn from a delta-lognormal distribution, for $i = 1, 2, ..., k$
⁸¹ and $j = 1, 2, .., n_i$. There are three parameters in this distribution: the mean μ_i, variance σ_i^2 and the
⁸² probability of obtaining a zero observation δ_i. The distribution of W_{ij} is given by

$$H(w_{ij}; \mu_i, \sigma_i^2, \delta_i) = \begin{cases} \delta_i & ; w_{ij} = 0 \\ \delta_i + (1 - \delta_i)G(w_{ij}; \mu_i, \sigma_i^2) & ; w_{ij} > 0 \end{cases} \tag{1}$$

where $G(w_{ij}; \mu_i, \sigma_i^2)$ is a lognormal distribution function, denoted as $LN(\mu_i, \sigma_i^2)$ such that $\ln W_{ij} \sim N(\mu_i, \sigma_i^2)$. The number of zeros has a binomial distribution $n_{i(0)} = \#\{j : w_{ij} = 0\} \sim B(n_i, \delta_i)$. The population mean of W_{ij} is given by

$$\vartheta_i = (1 - \delta_i) \exp(\mu_i + \frac{\sigma_i^2}{2}) \tag{2}$$

⁸³ The unbiased estimates of μ_i, σ_i^2, and δ_i are $\hat{\mu}_i = n_{i(1)}^{-1} \sum_{j:w_{ij}>0} \ln W_{ij}$, $\hat{\sigma}_i^2 = (n_{i(1)} - 1)^{-1} \sum_{j:w_{ij}>0} [\ln W_{ij} - \hat{\mu}_i]^2$,
⁸⁴ and $\hat{\delta}_i = n_{i(0)}/n_i$, respectively, where $n_i = n_{i(0)} + n_{i(1)}$; $n_{i(1)} = \#\{j : w_{ij} > 0\}$. Suppose that the delta-
⁸⁵ lognormal mean in (2) for all k populations are the same, then according to Tian and Wu (2007) and
⁸⁶ Krishnamoorthy and Oral (2015), the common delta-lognormal mean is defined as

$$\vartheta = (1-\delta_i)\exp(\mu_i + \frac{\sigma_i^2}{2}) \tag{3}$$

87 For the ith sample, the estimates of ϑ_i are $\hat{\vartheta}_i^* = (1-\hat{\delta}_i)\exp(\hat{\mu}_i + \frac{\hat{\sigma}_i^2}{2})$ which contains the unbiased estimates

88 $\hat{\mu}_i$, $\hat{\sigma}_i^2$ and $\hat{\delta}_i$. According to Longford (2009), the expected value of $\hat{\vartheta}_i^*$ is derived as

$$E[\hat{\vartheta}_i^*] = \left[1 - E(\hat{\delta}_i)\right] E\left[\exp\left\{\hat{\mu}_i + \frac{\hat{\sigma}_i^2}{2}\right\}\right] \tag{4}$$

$$= (1-\delta_i)\exp\left(\mu_i + \frac{\sigma_i^2}{n_{i(1)}}\right)\left(\frac{l_i}{l_i - \sigma_i^2}\right)^{l_i/2} \tag{5}$$

89 where $\hat{\delta}_i \sim N(\delta_i, \frac{\delta_i(1-\delta_i)}{n_i})$ as $n_i \to \infty$, $E[\exp(\hat{\mu}_i)] = \exp\left(\mu_i + \frac{\sigma_i^2}{2n_{i(1)}}\right)$ and $E[\exp(c_i Y_i)] = (1-2c_i)^{-l/2}$;

90 $Y_i = l_i\frac{\hat{\sigma}_i^2}{\sigma_i^2} \sim \chi_{l_i}^2$ and $c_i = \frac{\sigma_i^2}{2l_i}$, $\hat{\sigma}_i^2 = (n_{i(1)}-1)^{-1}\sum_{j=1}^{n_{i(1)}}\left[\ln(W_{ij}) - \hat{\mu}_i\right]^2$. If $\frac{l_i - \sigma_i^2}{l_i} = \exp\left[\frac{-2\sigma_i^2}{l_i}(\frac{1}{2} - \frac{1}{2n_{i(1)}})\right]$, then

91 we can obtain that

$$E[\hat{\vartheta}_i^*] = (1-\delta_i)\exp\left(\mu_i + \frac{\sigma_i^2}{2n_{i(1)}}\right)\left\{\exp\left[\frac{-2\sigma_i^2}{l_i}(\frac{1}{2} - \frac{1}{2n_{i(1)}})\right]\right\}^{-l_i/2}$$

$$= (1-\delta_i)\exp\left(\mu_i + \frac{\sigma_i^2}{2}\right) \tag{6}$$

92 According to Aitchison and Brown (1963), the Aitchison estimate of ϑ_i is expressed as

$$\hat{\vartheta}_i^{(Ait)} = \begin{cases} 0 & ; n_{i(1)} = 0 \\ w_{i1}/n_i & ; n_{i(1)} = 1 \\ (1-\hat{\delta}_i)\exp(\hat{\mu}_i)\psi_{n_{i(1)}}(\frac{\hat{\sigma}_i^2}{2}) & ; n_{i(1)} > 1 \end{cases} \tag{7}$$

93 where $\psi_a(b)$ is a Bessel function defined as

$$\psi_a(b) = 1 + \frac{(a-1)b}{a} + \frac{(a-1)^3}{a^2 2!}\frac{b^2}{a+1} + \frac{(a-1)^5}{a^3 3!}\frac{b^3}{(a+1)(a+3)} + \cdots \tag{8}$$

94 To investigate the unbiased estimate $\hat{\vartheta}_i^{(Ait)}$, the expected value is

$$E\left[\hat{\vartheta}_i^{(Ait)}\right] = \sum_{j=1}^{n_i} P(n_{i(1)} = j)E\left[\hat{\vartheta}_i | n_{i(1)} = j\right]$$

$$= 0 + P(n_{i(1)} = 1)E\left[w_{i1}/n_i\right] + \sum_{j=2}^{n_i} P(n_{i(1)} = j)E\left[\hat{\vartheta}_i | n_{i(1)} = j\right]$$

$$= P(n_{i(1)} = 1)\frac{\exp(\mu_i + \frac{\sigma_i^2}{2})}{n_i} + \sum_{j=2}^{n_i} P(n_{i(1)} = j)E\left[\frac{n_{i(1)}}{n_i}\exp(\mu_i + \frac{\sigma_i^2}{2})|n_{i(1)} = j\right]$$

$$= \sum_{j=0}^{n_i} P(n_{i(1)} = j)E\left[\frac{n_{i(1)}}{n_i}\exp(\mu_i + \frac{\sigma_i^2}{2})|n_{i(1)} = j\right]$$

$$= E\left[\frac{n_{i(1)}}{n_i}\exp(\mu_i + \frac{\sigma_i^2}{2})\right]$$

$$= (1-\delta_i)\exp(\mu_i + \frac{\sigma_i^2}{2})$$

According to Shimizu and Iwase (1981), the uniformly minimum variance unbiased (UMVU) estimate of ϑ_i is

$$\hat{\vartheta}_i^{(Shi)} = \begin{cases} 0 & ; n_{i(1)} < 1 \\ \frac{n_{i(1)}}{n_i} \exp(\hat{\mu}_i)_0F_1\left(\frac{n_{i(1)}-1}{2}; \frac{n_{i(1)}-1}{4n_{i(1)}}S_i^2\right) & ; n_{i(1)} \geq 1 \end{cases} \tag{9}$$

where $S_i^2 = \sum_{j=1}^{n_{i(1)}}\left[\ln(W_{ij}) - \hat{\mu}_i\right]^2$ and $_0F_1(a;z) = \sum_{m=0}^{\infty}\frac{z^m}{(a)_m m!}$;

$$(a)_m = \begin{cases} 1 & ; m = 0 \\ a(a+1)...(a+m-1) & ; m \geq 1 \end{cases} \tag{10}$$

95 From Kunio (1983), $E\left[_0F_1\left(\frac{n_{i(1)}-1}{2}; \frac{a}{2}S_i^2\right)\right] = \exp(a\sigma^2)$ is obtained, then

$$\begin{aligned} E\left[\hat{\vartheta}_i^{(Shi)}\right] &= E\left[\frac{n_{i(1)}}{n}\exp(\hat{\mu}_i)_0F_1\left(\frac{n_{i(1)}-1}{2}, \frac{n_{i(1)}-1}{4n_{i(1)}}S_i^2\right)\right] \\ &= \frac{n_i(1-\delta_i)}{n_i}\exp\left[\mu_i + \frac{\sigma_i^2}{2n_{i(1)}}\right]\exp\left[\frac{n_{i(1)}-1}{2n_{i(1)}}\sigma_i^2\right] \\ &= (1-\delta_i)\exp\left(\mu_i + \frac{\sigma_i^2}{2}\right) \end{aligned} \tag{11}$$

96 where $E(n_{i(1)}) = n_i(1-\delta_i)$. The asymptotic variance of $\hat{\vartheta}_i^{(Shi)}$ is given by

$$\begin{aligned} Var\left[\hat{\vartheta}_i^{(Shi)}\right] &= \exp(2\mu_i + \sigma_i^2)\left[\frac{1}{n_i^2}\sum_{j=1}^{n_i}\binom{n_i}{j}(1-\delta_i)^j\delta^{n_i-j}j^2\exp\left(\frac{\sigma_i^2}{j}\right)\right. \\ &\quad \left._0F_1\left(\frac{j-1}{2}; \frac{(j-1)^2}{4j^2}\sigma_i^4\right) - (1-\delta_i)^2\right] \\ &= \frac{\exp(2\mu_i + \sigma_i^2)}{n_i}\left[\delta_i(1-\delta_i) + \frac{1}{2}(1-\delta_i)(2\sigma_i^2 + \sigma_i^4)\right] + O(n^{-2}) \end{aligned} \tag{12}$$

97 Actually, $\psi_{n_{i(1)}}(\frac{\hat{\sigma}_i^2}{2}) = {}_0F_1\left(\frac{n_{i(1)}-1}{2}; \frac{n_{i(1)}-1}{4n_{i(1)}}S_i^2\right)$ such that $\hat{\vartheta}_i^{(Shi)}$ and $\hat{\vartheta}_i^{(Ait)}$ are the unbiased estimates of ϑ_i
98 under different ideas, although their variances are the same i.e. $Var\left[\hat{\vartheta}_i^{(Shi)}\right] = Var\left[\hat{\vartheta}_i^{(Ait)}\right]$. Using $\hat{\mu}_i, \hat{\sigma}_i^2$,
99 and $\hat{\delta}_i$ from the samples, the estimated delta-lognormal mean $\hat{\vartheta}_i^{(Ait)}$ and variance of $\hat{\vartheta}_i^{(Ait)}$ are obtained.
100 The following methods are the detailed construction of the CIs for the common delta-lognormal mean.

101 **Fiducial Generalized Confidence Interval**
102 Fiducial inference was introduced by Fisher (1930). Fisher's fiducial argument was used to develop a
103 generalized fiducial recipe that could be extended to the application of fiducial ideas (Hannig, 2009). The
104 concept of the fiducial interval has been advanced by the idea of the generalized pivotal quantity (GPQ)
105 such that it is directly used to apply for generalized inference. Later, Hannig et al. (2006) argued that a
106 subclass of GPQs, the fiducial GPQ (FGPQ), provides a framework that shows the connection between
107 a distribution and a parameter. Recall that $\hat{\mu}_i \sim N(\mu_i, \sigma_i^2/n_{i(1)})$ and $(n_{i(1)}-1)\hat{\sigma}_i^2/\sigma_i^2 \sim \chi_{n_{i(1)}-1}^2$ are the
108 independent random variables. The structure functions of $\hat{\mu}_i$ and $\hat{\sigma}_i^2$ are

$$\hat{\mu}_i = \mu_i + V_i\sqrt{\frac{\sigma_i^2}{n_{i(1)}}} \quad \text{and} \quad \hat{\sigma}_i^2 = \frac{\sigma_i^2 U_i}{n_{i(1)}-1} \tag{13}$$

109 which are the function of V_i and U_i, respectively, where $V_i \sim N(0,1)$ and $U_i \sim \chi_{n_{i(1)}-1}^2$. Given the
110 observed values, the estimates $\hat{\mu}_i$ and $\hat{\sigma}_i^2$ can be obtained, and the unique solution of $(\hat{\mu}_i, \hat{\sigma}_i^2) =$
111 $\left(\mu_i + V_i\sqrt{\frac{\sigma_i^2}{n_{i(1)}}}, \frac{\sigma_i^2 U_i}{n_{i(1)}-1}\right)$ becomes

$$\mu_i = \hat{\mu}_i - V_i \frac{\hat{\sigma}_i}{\sqrt{n_{i(1)}}} \sqrt{\frac{n_{i(1)} - 1}{U_i}}, \qquad \sigma_i^2 = \frac{(n_{i(1)} - 1)\hat{\sigma}_i^2}{U_i} \qquad (14)$$

112 The respective FGPQs of μ_i and σ_i^2 are

$$G_{\mu_i} = \hat{\mu}_i - V_i^* \frac{\hat{\sigma}_i}{\sqrt{n_{i(1)}}} \sqrt{\frac{n_{i(1)} - 1}{U_i^*}} \qquad (15)$$

$$G_{\sigma_i^2} = \frac{(n_{i(1)} - 1)\hat{\sigma}_i^2}{U_i^*} \qquad (16)$$

113 where V_i^* and U_i^* are independent copies of V_i and U_i, respectively. Hasan and Krishnamoorthy (2018)
114 developed the FGPQ of δ_i using a beta distribution as $G_{\delta_i'} \sim Beta(\alpha_i, \beta_i)$; $\alpha_i = n_{i(1)} + 0.5$ and $\beta_i = n_{i(0)} + 0.5$.
115 The FGPQ of ϑ based on k individual samples is

$$G_\vartheta = \frac{\sum_{i=1}^k G_{w_i} G_{\vartheta_i}}{\sum_{i=1}^k G_{w_i}} \qquad (17)$$

116 where $G_{\vartheta_i} = G_{\delta_i'} \exp(G_{\mu_i} + G_{\sigma_i^2}/2)$, $G_{w_i} = 1/G_{Var\left[\hat{\vartheta}_i^{(Ait)}\right]}$, and

117 $G_{Var\left[\hat{\vartheta}_i^{(Ait)}\right]} = \exp(2G_{\mu_i} + G_{\sigma_i^2})\left[G_{\delta_i'}(1 - G_{\delta_i'}) + \frac{1}{2}G_{\delta_i'}(2G_{\sigma_i^2} + G_{\sigma_i^4})\right]/n_i$. Thus, the 100(1- ζ)%FGCI for
118 ϑ is

$$CI_\vartheta^{(fgci)} = [L_\vartheta^{(fgci)}, U_\vartheta^{(fgci)}] = [G_\vartheta(\zeta/2), G_\vartheta(1 - \zeta/2)] \qquad (18)$$

119 where $G_\vartheta(\zeta)$ denotes the ζth percentiles of G_ϑ. Algorithm 1 shows the computational steps for obtaining
120 the FGCI.

121 ***Algorithm 1: FGCI***
122   1) Generate $V_i \sim N(0,1)$ and $U_i \sim \chi_{n_{i(1)}-1}^2$ are independent.
123   2) Compute the FGPQs G_{μ_i}, $G_{\sigma_i^2}$ and $G_{\delta_i'}$.
124   3) Compute G_{w_i} and G_{ϑ_i} leading to obtain G_ϑ.
125   4) Repeat steps 1-3, a number of times, m=2500, compute 95%FGCI for ϑ, as given in (18).

126 **Large Sample Interval**
127 Recall that the Aitchitson estimator is $\hat{\vartheta}_i^{(Ait)} = (1 - \hat{\delta}_i)\exp(\hat{\mu}_i)\psi_{n_{i(1)}}(\hat{\sigma}_i^2/2)$ and the variance of $\hat{\vartheta}_i^{(Ait)}$
128 is $Var\left[\hat{\vartheta}_i^{(Ait)}\right] = \exp(2\mu_i + \sigma_i^2)\left[\delta_i(1 - \delta_i) + \frac{1}{2}(1 - \delta_i)(2\sigma_i^2 + \sigma_i^4)\right]/n_i$. The approximated variance is ob-
129 tained by replacing $\hat{\mu}_i$, $\hat{\sigma}_i^2$ and $\hat{\delta}_i$. The pooled estimate of ϑ_i is given by

$$\hat{\vartheta} = \frac{\sum_{i=1}^k w_i \hat{\vartheta}_i^{(Ait)}}{\sum_{i=1}^k w_i} \qquad (19)$$

130 where $w_i = 1/\widehat{Var}\left[\hat{\vartheta}_i^{(Ait)}\right]$. Hence, the $100(1 - \zeta)\%$ LS interval for ϑ is obtained as

$$CI_\vartheta^{(ls)} = [L_\vartheta^{(ls)}, U_\vartheta^{(ls)}] = \left[\hat{\vartheta} - z_{1-\frac{\zeta}{2}}\sqrt{1/\sum_{i=1}^k w_i}, \hat{\vartheta} + z_{1-\frac{\zeta}{2}}\sqrt{1/\sum_{i=1}^k w_i}\right] \qquad (20)$$

131 where z_ζ denotes the ζth percentiles of standard normal $N(0,1)$. The LS interval can be estimated easily
132 via Algorithm 2.

Algorithm 2: LS

1) Compute $\hat{\vartheta}_i^{(Ait)}$ and $\widehat{Var}\left[\hat{\vartheta}_i^{(Ait)}\right]$.

2) Compute $\hat{\vartheta}$.

3) Compute 95%LS interval for ϑ, as given in (20).

Method of Variance Estimates Recovery

This method produces a closed-form CI that is easy to compute. For this reason, the MOVER CI for the common delta-lognormal mean is considered for k individual random samples. The MOVER for a linear combination of $\vartheta_i; i = 1, 2, ..., k$ is as follows. Let $\hat{\vartheta}_1, \hat{\vartheta}_2, ..., \hat{\vartheta}_k$ be independent unbiased estimators of $\vartheta_1, \vartheta_2, ..., \vartheta_k$, respectively. In addition, let $[l_i, u_i]$ stand for the $100(1-\zeta)\%$CI for ϑ_i. According to Krishnamoorthy and Oral (2015), the $100(1-\zeta)\%$MOVER for $\sum_{i=1}^{k} c_i \vartheta_i$ is given by

$$CI_{\sum_{i=1}^{k} c_i \vartheta_i} = \left[L_{\sum_{i=1}^{k} c_i \vartheta_i}, U_{\sum_{i=1}^{k} c_i \vartheta_i}\right]$$

$$= \left[\sum_{i=1}^{k} c_i \hat{\vartheta}_i - \sqrt{\sum_{i=1}^{k} c_i^2 \left(\hat{\vartheta}_i - l_i^*\right)^2}, \sum_{i=1}^{k} c_i \hat{\vartheta}_i + \sqrt{\sum_{i=1}^{k} c_i^2 \left(\hat{\vartheta}_i - u_i^*\right)^2}\right] \tag{21}$$

where $l_i^* = \begin{cases} l_i & ; c_i > 0 \\ u_i & ; c_i < 0 \end{cases}$ and $u_i^* = \begin{cases} u_i & ; c_i > 0 \\ l_i & ; c_i < 0 \end{cases}$. Next, the closed-form CIs for ϑ_i are needed to construct MOVER for ϑ. Thus, ϑ_i is log-transformed as

$$\ln \vartheta_i = \ln \delta_i^* + (\mu_i + \sigma_i^2) \tag{22}$$

where $\delta_i^* = 1 - \delta_i$. Let $\hat{\mu}_i$, and $\hat{\sigma}_i^2$ and $\hat{\delta}^*$ be the unbiased estimates of μ_i, σ_i^2, and δ_i, respectively. The MOVER for a single delta-lognormal mean presented by Hasan and Krishnamoorthy (2018), the MOVER for ϑ_i is given by

$$L_{\vartheta_i} = \exp\left\{\ln \hat{\delta}_i^* + (\hat{\mu}_i + \hat{\sigma}_i^2) - \sqrt{\left(\ln \hat{\delta}_i^* - l_{\ln \delta_i^*}\right)^2 + \left(\hat{\mu}_i + \hat{\sigma}_i^2 - l_{\mu_i + \sigma_i^2}\right)^2}\right\}$$

$$U_{\vartheta_i} = \exp\left\{\ln \hat{\delta}_i^* + (\hat{\mu}_i + \hat{\sigma}_i^2) - \sqrt{\left(\ln \hat{\delta}_i^* - u_{\ln \delta_i^*}\right)^2 + \left(\hat{\mu}_i + \hat{\sigma}_i^2 - u_{\mu_i + \sigma_i^2}\right)^2}\right\} \tag{23}$$

where

$$(l_{\ln \delta_i^*}, u_{\ln \delta_i^*}) = \ln\left[\left(\hat{\delta}_i^* + \frac{T_{i,\zeta/2}^2}{2n_i} \mp T_{i,1-\zeta/2}\sqrt{\frac{\hat{\delta}_i^*(1-\hat{\delta}_i^*)}{n_i} + \frac{T_{i,\zeta/2}^2}{4n_i^2}}\right)\Big/(1 + T_{i,\zeta/2}^2/n_i)\right]$$

$$(l_{\mu_i + \sigma_i^2}, u_{\mu_i + \sigma_i^2}) = \left[(\hat{\mu}_i + \hat{\sigma}_i^2/2) - \left\{\left(\frac{Z_{i,\zeta/2}\hat{\sigma}_i^2}{n_{i(1)}}\right)^2 + \frac{\hat{\sigma}_i^4}{4}\left(1 - \frac{n_{i(1)} - 1}{\chi_{i,1-\zeta/2,n_{i(1)}-1}^2}\right)^2\right\}^{1/2},\right. \tag{24}$$

$$\left.(\hat{\mu}_i + \hat{\sigma}_i^2/2) + \left\{\left(\frac{Z_{i,\zeta/2}\hat{\sigma}_i^2}{n_{i(1)}}\right)^2 + \frac{\hat{\sigma}_i^4}{4}\left(\frac{n_{i(1)} - 1}{\chi_{i,\zeta/2,n_{i(1)}-1}^2} - 1\right)^2\right\}^{1/2}\right]$$

Note that both $T_i = (n_{i(1)} - n_i \delta^*)/\sqrt{n_i \delta_i^*(1-\delta_i^*)} \overset{d}{\sim} N(0,1)$, and $Z_i = (\hat{\mu}_i - \mu_i)/\sqrt{\hat{\sigma}_i^2/n_{i(1)}} \overset{d}{\sim} N(0,1)$ are independent random variables. According to Krishnamoorthy and Oral (2015), the $100(1-\zeta)\%$ MOVER interval for ϑ is

$$CI_\vartheta^{(mover)} = [L_\vartheta, U_\vartheta]$$

$$= \left[\frac{\sum_{i=1}^k w_i \hat{\vartheta}_i^{(Ait)}}{\sum_{i=1}^k w_i} - \sqrt{\frac{\sum_{i=1}^k w_i^2 \left(\hat{\vartheta}_i^{(Ait)} - L_{\vartheta_i} \right)^2}{\sum_{i=1}^k w_i^2}}, \frac{\sum_{i=1}^k w_i \hat{\vartheta}_i^{(Ait)}}{\sum_{i=1}^k w_i} - \sqrt{\frac{\sum_{i=1}^k w_i^2 \left(\hat{\vartheta}_i^{(Ait)} - U_{\vartheta_i} \right)^2}{\sum_{i=1}^k w_i^2}} \right]$$

(25)

152   where $w_i = 1/\widehat{Var}\left[\hat{\vartheta}_i^{(Ait)} \right]$. Algorithm 3 describes the steps to construct the MOVER interval.

153   ### *Algorithm 3: MOVER*
154   1) Compute CIs for $\ln \delta_i^*$ and $\mu_i + \sigma_i^2$ are $(l_{\ln \delta_i^*}, u_{\ln \delta_i^*})$ and $(l_{\mu_i + \sigma_i^2}, u_{\mu_i + \sigma_i^2})$, respectively.
155   2) Compute MOVER for ϑ_i, as given in (23).
156   3) Compute 95%MOVER for ϑ, given in (25).

157   ### **Parametric Bootstrap**
158   This is developed from the parametric bootstrap on the common mean of several heterogeneous log-
159   normal distributions, proposed by Malekzadeh and Kharrati-Kopaei (2019). The delta-lognormal mean is
160   transformed by taking the logarithm as

$$\mu_i = \ln\left(\frac{\vartheta}{1 - \delta_i} \right) - \frac{\sigma_i^2}{2}$$

(26)

161   The likelihood of $(\vartheta, \sigma_i^2, \delta_i)$ is

$$L(\vartheta, \sigma_i^2, \delta_i | w_{ij}) = \prod_{i=1}^k \binom{n_i}{n_{i(0)}} \delta_i (1 - \delta_i) \frac{1}{(2\pi\sigma_i^2)^{n_{i(1)}/2}} \exp\left\{ -\frac{1}{2\sigma_i^2} \sum_{j=1}^{n_{i(1)}} \left(\ln w_{ij} - \ln\left(\frac{\vartheta}{1 - \delta_i} \right) + \frac{\sigma_i^2}{2} \right)^2 \right\}$$

(27)

162   which enables obtaining the maximum likelihood estimates of $\ln \vartheta$ and σ_i^2 as

$$\ln \hat{\vartheta}_{mle} = \frac{\sum_{i=1}^k \hat{w}_{mle,i} \left[\hat{\mu}_i + \ln(1 - \hat{\delta}_i) \right] + N/2}{\sum_{i=1}^k \hat{w}_{mle,i}}$$

$$\hat{\sigma}_{mle,i}^2 = -2 + 2\sqrt{1 + \hat{\sigma}_i^2 + \{\hat{\mu} - \ln[\hat{\vartheta}/(1 - \hat{\delta}_i)]\}^2}$$

(28)

163   where $\hat{w}_{mle,i} = n_{i(1)}/\hat{\sigma}_{mle,i}^2$ and $\ln \hat{\vartheta} = \frac{\sum_{i=1}^k \hat{w}_i \left[\hat{\mu}_i + \ln(1 - \hat{\delta}_i) \right] + N/2}{\sum_{i=1}^k \hat{w}_i}$; $\hat{w}_i = n_{i(1)}/\hat{\sigma}_i^2$. If $\delta_i = 0$, then it becomes the
164   common lognormal mean (see Krishnamoorthy and Oral (2015) for a detailed explanation). By applying
165   central limit theorem, we obtain $\left(\ln \hat{\vartheta}_{mle} - \ln \vartheta \right) \sqrt{\sum_{i=1}^k \hat{w}_{mle,i}} \sim N(0,1)$ such that $T = \left(\ln \hat{\vartheta}_{mle} - \ln \vartheta \right)^2 \sum_{i=1}^k \hat{w}_{mle,i} \sim$
166    $\chi_{n_{i(1)}-1}^2$. It is well-known that $\hat{\mu}_i$, $\hat{\sigma}_i^2$ and $\hat{\delta}_i$ are independent random variables for which $\hat{\mu}_i \sim N(\ln\left(\frac{\vartheta}{1 - \delta_i} \right) -$
167    $\frac{\sigma_i^2}{2}, \sigma_i^2/n_{i(1)})$, $(n_{i(1)} - 1)\hat{\sigma}_i^2/\sigma_i^2 \sim \chi_{n_{i(1)}-1}^2$ and $\hat{\delta}_i \sim N(\delta, \delta(1 - \delta)/n_i)$ are obtained, respectively. Let
168    $\eta = \mu_i + \sigma_i^2/2$ so that we can write $T = \frac{\sum_{i=1}^k \hat{w}_{mle,i} \left[\hat{\mu}_i + \ln(1 - \hat{\delta}_i) - \eta - \ln(1 - \delta_i) \right] + N/2}{\sum_{i=1}^k \hat{w}_{mle,i}}$. It can be seen that the dis-
169   tribution of T is complicated, possibly depending on nuisance parameters σ_i^2 and δ_i, but not on $\ln \vartheta$.
170   Thus, the exact distribution of T is unknown in practice, and so we propose the PB pivotal variable
171   corresponding to T^{PB} as

$$T^{PB} = \left(\ln \hat{\vartheta}_{mle}^{PB} - \ln \hat{\vartheta} \right)^2 \sum_{i=1}^k \hat{w}_{mle,i}^{PB}$$

(29)

where $\ln \hat{\vartheta}_{mle}^{PB} = \frac{\sum_{i=1}^{k} \hat{w}_{mle,i}^{PB}\left[\hat{\mu}_i^{PB}+\ln(1-\hat{\delta}_i^{PB})\right]+N/2}{\sum_{i=1}^{k} \hat{w}_{mle,i}^{PB}}$, $\hat{w}_i^{PB} = n_{i(1)}/\hat{\sigma}_i^{2B}$, $\hat{\mu}_i^{PB} \sim N(\hat{\mu}_i^B, \hat{\sigma}_i^{2B}/n_{i(1)})$, $\hat{\sigma}_i^{2PB} \sim \hat{\sigma}_i^{B2}\chi_{n_{i(1)}-1}^2$

$/(n_{i(1)}-1)$ and $\hat{\delta}^{PB} \sim beta(n_{i(0)}^B+0.5, n_{i(1)}^B+0.5)$, $n_{i(0)}^B = n_i\hat{\delta}_i^B$, and $n_{i(1)}^B = n_i - n_{i(0)}^B$. Note that $\hat{\mu}_i^B$, $\hat{\sigma}_i^{2B}$, and $\hat{\delta}_i^B$ are the observed values of $\hat{\mu}_i$, $\hat{\sigma}_i^2$, and $\hat{\delta}_i$, respectively, from random sampling with replacement based on the bootstrap approach. Thus, the $100(1-\zeta)\%$PB interval for ϑ is given by

$$CI_{\vartheta}^{(pb)} = \exp\left[\ln\hat{\vartheta}_{mle} \mp \sqrt{q_{\zeta}^{PB}/\sum_{i=1}^{k}\hat{w}_{mle,i}}\right] \tag{30}$$

where q_{ζ}^{PB} denotes the $(1-\zeta)^{th}$ percentile of distribution of T^{PB}. The PB interval can be constructed as shown in Algorithm 4.

Algorithm 4: PB

1) Compute $\hat{\mu}_i$, $\hat{\sigma}_i^2$ and $\hat{\delta}$ leading to obtain $\ln\hat{\vartheta}$.
2) Compute $\ln\hat{\vartheta}_{mle}$ and $\hat{\sigma}_{mle,i}^2$.
3) Generate $\hat{\mu}_i^{PB}$, $\hat{\sigma}_i^{2PB}$ and $\hat{\delta}_i^{PB}$ leading to compute $\ln\hat{\vartheta}_{mle}^{PB}$.
4) Repeat steps 1-3, a number of time $m = 2500$, compute T^{PB} to obtain q_{ζ}^{PB}.
5) Compute 95%PB interval for ϑ, as given in (30).

Highest Posterior Density Intervals

The HPD interval is constructed from the posterior distribution, as defined by Box and Tiao (1973). Note that the prior of ϑ_i is updated with its likelihood function thereby obtaining the posterior distribution based on the Bayesian approach. Recall that $W_{ij} \sim \Delta(\mu_i, \sigma_i^2, \delta_i)$, then the likelihood is given by

$$P(w_{ij}|\mu_i, \sigma_i^2, \delta_i) \propto \prod_{i=1}^{k} \delta_i^{n_{i(0)}} (1-\delta_i)^{n_{i(1)}} (\sigma_i^2)^{-n_{i(1)}/2}\exp\left\{-\frac{1}{2\sigma_i^2}\sum_{j=1}^{n_{i(1)}}\left(\ln w_{ij}-\mu_i\right)^2\right\} \tag{31}$$

For k individual samples, Miroshnikov et al. (2015) described the pooled independent sub-posterior samples toward the joint posterior distributions ϑ are combined using weighted averages as follows:

$$\vartheta^{post} = \sum_{i=1}^{k} w_i\vartheta_i^{post}\left(\sum_{i=1}^{k}w_i\right)^{-1} \tag{32}$$

where ϑ_i^{post} are the posterior samples of ϑ_i, for $i = 1,2,...,k$. The inverse of the sample variance is used to weight the posterior based on the ith samples is denoted as $w_i = Var^{-1}(\hat{\vartheta}_i|w_{ij})$. Different priors have been developed for estimating the common delta-lognormal mean, two of which are derived in the following subsections.

Jeffreys' Rule Prior

Harvey and van der Merwe (2012) defined this prior as

$$P(\vartheta)_{JR} \propto \prod_{i=1}^{k} \sigma_i^{-3}\delta_i^{-1/2}(1-\delta_i)^{1/2} \tag{33}$$

which is combined with the likelihood (34) to obtain the posterior of ϑ as

$$\begin{aligned}
P(w_{ij}|\vartheta) &\propto \prod_{i=1}^{k} \delta_i^{n_{i(0)}-1/2}(1-\delta_i)^{n_{i(1)}+1/2}(\sigma_i^2)^{-(n_{i(1)}+3)/2}\exp\left\{-\frac{1}{2\sigma_i^2}\sum_{j=1}^{n_{i(1)}}\left(\ln w_{ij}-\mu_i\right)^2\right\} \\
&\propto \prod_{i=1}^{k} \delta_i^{(n_{i(0)}+1/2)-1}(1-\delta_i)^{(n_{i(1)}+3/2)-1}(\sigma_i^2)^{-\frac{(n_{i(1)}+1)}{2}-1} \\
&\quad \exp\left\{-\frac{1}{2\sigma_i^2}\left[(n_{i(1)}-1)\hat{\sigma}_i^2+n_{i(1)}(\hat{\mu}_i-\mu_i)^2\right]\right\}
\end{aligned} \tag{34}$$

197 This leads to obtaining the marginal posterior distributions of μ_i, σ_i^2 and δ_i as

$$\mu_i^{(JR)}|\sigma_{i,JR}^2, w_{ij} \sim N\left(\hat{\mu}_i, \sigma_i^{2(JR)}/n_{i(1)}\right)$$
$$\sigma_i^{2(JR)}|w_{ij} \sim IG\left((n_{i(1)}+1)/2, (n_{i(1)}+1)\hat{\sigma}_i^2/2\right) \tag{35}$$
$$\delta_i^{(JR)}|w_{ij} \sim beta\left(n_{i(0)}+1/2, n_{i(1)}+3/2\right)$$

198 The pooled posterior of ϑ is weighted by its inversely estimated variance as follows:

$$\vartheta^{post} = \sum_{i=1}^{k} w_i^{(JR)} \vartheta_i^{(JR)p} \left(\sum_{i=1}^{k} w_i^{(JR)}\right)^{-1} \tag{36}$$

199 where
200 $\quad \vartheta_i^{(JR)p} = (1-\delta_i^{(JR)})\exp(\mu_i^{(JR)} + \sigma_i^{2(JR)}/2)$
201 $\quad w_i^{(JR)} = \left\{n_i^{-1}\exp(2\mu_i^{(JR)} + \sigma_i^{2(JR)})\left[\delta_i^{(JR)}(1-\delta_i^{(JR)}) + \frac{1}{2}(1-\delta_i^{(JR)})(2\sigma_i^{2(JR)} + \sigma_i^{4(JR)})\right]\right\}^{-1}$
202 From (36), the $100(1-\zeta)\%$HPD-based Jeffreys' rule prior (HPD-JR) for ϑ is constructed as follows:

203 *Normal-Gamma-Beta Prior*
204 Maneerat et al. (2020) proposed a HPD based on the normal-gamma prior for the ratio of delta-lognormal
205 variances that worked better than the HPD-JR of Harvey and van der Merwe (2012). Suppose that
206 $Y = \ln W$ be a random variable of normal distribution with mean $\mu = (\mu_1, \mu_2, ..., \mu_k)$ and precision
207 $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ where $W \sim LN(\mu, \lambda)$ and $\lambda_i = \sigma_i^{-2}$. The HPD-based normal-gamma-beta prior (HPD-
208 NGB) of $\vartheta = (\mu_i, \lambda_i, \delta_i)'$ is defined as

$$P(\vartheta) \propto \prod_{i=1}^{k} \lambda_i^{-1}[\delta_i(1-\delta_i)]^{-1/2} \tag{37}$$

209 where (μ_i, λ_i) follows a normal-gamma distribution, and δ_i follows a beta distribution, denoted as
210 $(\mu_i, \lambda_i) \sim NG(\mu_i, \lambda_i|\mu, k_{i(0)}=0, \alpha_{i(0)}=-1/2, \beta_{i(0)}=0)$ and $\delta_i \sim beta(1/2, 1/2)$, respectively. When the
211 the prior (37) is combined with the likelihood (34), then the posterior density of ϑ becomes

$$P(\vartheta|w_{ij}) \propto \prod_{i=1}^{k} \delta_i^{n_{i(0)}-1/2}(1-\delta_i)^{n_{i(1)}-1/2}\lambda_i^{\frac{n_{i(1)}-1}{2}-1}\exp\left\{-\frac{\lambda_i}{2}\sum_{j=1}^{n_{i(1)}}(\ln w_{ij}-\hat{\mu}_i)^2\right\}\lambda_i^{1/2}$$
$$\exp\left\{-\frac{n_{i(1)}\lambda_i}{2}(\mu_i-\mu_i^*)^2\right\} \tag{38}$$

212 which can be integrated out to obtain the marginal posterior distributions of μ_i, λ_i and δ_i as follows:

$$\mu_i^{(NGB)}|w_{ij} \sim t_{df}\left(\mu_i|\hat{\mu}_i, \sum_{j=1}^{n_{i(1)}}(\ln w_{ij}-\hat{\mu}_i)^2/[n_{i(1)}(n_{i(1)}-1)]\right)$$
$$\lambda_i^{(NGB)}|w_{ij} \sim G\left(\lambda_i|(n_{i(1)}-1)/2, \sum_{j=1}^{n_{i(1)}}(\ln w_{ij}-\hat{\mu}_i)^2/2\right) \tag{39}$$
$$\delta_i^{(NGB)}|w_{ij} \sim beta\left(n_{i(0)}+1/2, n_{i(1)}+1/2\right)$$

213 where $df = 2(n_{i(1)}-1)$ and $\sigma_i^{2(NGB)}|w_{ij} \sim IG\left(\sigma_i^2|(n_{i(1)}-1)/2, \sum_{j=1}^{n_{i(1)}}(\ln w_{ij}-\hat{\mu}_i)^2/2\right)$. Similarly, the
214 pooled posterior of ϑ is given by

$$\vartheta^{post} = \sum_{i=1}^{k} w_i^{(NGB)} \vartheta_i^{(NGB)p} \left(\sum_{i=1}^{k} w_i^{(NGB)}\right)^{-1} \tag{40}$$

where
$$\vartheta_i^{(NGB)p} = (1 - \delta_i^{(NGB)}) \exp(\mu_i^{(NGB)} + \sigma_i^{2(NGB)}/2)$$

$$w_i^{(NGB)} = \left\{ n_i^{-1} \exp(2\mu_i^{(NGB)} + \sigma_i^{2(NGB)}) \left[\delta_i^{(NGB)}(1 - \delta_i^{(NGB)})\tfrac{1}{2}(1 - \delta_i^{(NGB)})(2\sigma_i^{2(NGB)} + \sigma_i^{4(NGB)}) \right] \right\}^{-1}$$

Hence, the $100(1 - \zeta)\%$HPD-HGB for ϑ is constructed in (40). Algorithm 5 details the steps to construct the HPD-JR and HPD-NGB.

Algorithm 5: HPD-JR and HPD-NGB

1) Compute $\hat{\mu}_i$, $\hat{\sigma}_i^2$ and $\hat{\delta}$.
2) Generate the posterior densities of μ_i, σ_i^2 and δ_i based-Jeffreys' rule (JR) and normal-gamma-beta (NGB) priors, as given in (35) and (39), respectively.
3) Compute the pooled posterior of ϑ based on JR and NGB priors, as given in (36) and (40), respectively.
4) Compute 95%HPD-JR and HPD-NGB for ϑ, defined by Box and Tiao (1973).

SIMULATION STUDIES AND RESULTS

The performances of the CIs were assessed by comparing their coverage probabilities (CPs) and average length (ALs) using Monte Carlo simulation. The best-performing CI is the one where the CP is closest to or greater than the nominal confidence level $1 - \zeta$ while also having an AL with the narrowest width. The CIs for the common delta-lognormal mean constructed using FGCI, LS, MOVER, PB, HPD-JR, and HPD-NGB were assessed in the study, the parameter settings for which are provided in Table 1. The number of generated random samples was fixed at $M = 5000$. For FGCI, the number of FGPQs was $Q = 2500$ for each set of 5000 random samples. Algorithm 6 shows the computational steps to estimate the CP and AL performances of all of the methods.

Table 1. Parameter settings for sample cases $k = 2, 5, 10$.

Scenarios	$(n_1,...,n_k)$	$(\delta_1,...,\delta_k)$	$(\sigma_1^2,...,\sigma_k^2)$
$k = 2$			
1-9	(30_2)	$(0.1,0.2), (0.2,0.5), (0.3,0.7)$	$(1,2), (2,4), (3,5)$
10-18	$(30,50)$	$(0.1,0.2), (0.2,0.5), (0.3,0.7)$	$(1,2), (2,4), (3,5)$
19-27	(50_2)	$(0.1,0.2), (0.2,0.5), (0.3,0.7)$	$(1,2), (2,4), (3,5)$
28-36	$(50,100)$	$(0.1,0.2), (0.2,0.5), (0.3,0.7)$	$(1,2), (2,4), (3,5)$
37-45	(100_2)	$(0.1,0.2), (0.2,0.5), (0.3,0.7)$	$(1,2), (2,4), (3,5)$
$k = 5$			
46-54	(30_5)	$(0.05,0.1_2,0.2_2), (0.2_2,0.4_3), (0.5_2,0.7_3)$	$(1_2,2_3), (2_2,3_3), (3_2,5_3)$
55-63	$(30_2,50_3)$	$(0.05,0.1_2,0.2_2), (0.2_2,0.4_3), (0.5_2,0.7_3)$	$(1_2,2_3), (2_2,3_3), (3_2,5_3)$
64-72	$(30_2,50_2,100)$	$(0.05,0.1_2,0.2_2), (0.2_2,0.4_3), (0.5_2,0.7_3)$	$(1_2,2_3), (2_2,3_3), (3_2,5_3)$
73-81	$(30,50_2,100_2)$	$(0.05,0.1_2,0.2_2), (0.2_2,0.4_3), (0.5_2,0.7_3)$	$(1_2,2_3), (2_2,3_3), (3_2,5_3)$
82-90	(50_5)	$(0.05,0.1_2,0.2_2), (0.2_2,0.4_3), (0.5_2,0.7_3)$	$(1_2,2_3), (2_2,3_3), (3_2,5_3)$
91-99	$(50_2,100_3)$	$(0.05,0.1_2,0.2_2), (0.2_2,0.4_3), (0.5_2,0.7_3)$	$(1_2,2_3), (2_2,3_3), (3_2,5_3)$
100-108	(100_5)	$(0.05,0.1_2,0.2_2), (0.2_2,0.4_3), (0.5_2,0.7_3)$	$(1_2,2_3), (2_2,3_3), (3_2,5_3)$
$k = 10$			
109-114	$(30_5,50_5)$	$(0.1_5,0.2_5), (0.2_5,0.5_5)$	$(1_5,2_5), (2_5,4_5), (3_5,5_5)$
115-120	$(30_3,50_3,100_4)$	$(0.1_5,0.2_5), (0.2_5,0.5_5)$	$(1_5,2_5), (2_5,4_5), (3_5,5_5)$
121-126	$(50_5,100_5)$	$(0.1_5,0.2_5), (0.2_5,0.5_5)$	$(1_5,2_5), (2_5,4_5), (3_5,5_5)$

Note: (30_5) stands for $(30,30,30,30,30)$.

Algorithm 6: Comparison of CPs and ALs for all CIs.

For $g = 1$ to M. Generate $w_{ij} \sim \Delta(\mu_i, \sigma_i^2, \delta_i)$.

Compute the unbiased estimates $\hat{\mu}_i$, $\hat{\sigma}_i^2$ and $\hat{\delta}$.

Compute the 95%CIs for ϑ based on FGCI, LS, MOVER, PB and the HPDs via Algorithms 1, 2, 3, 4 and 5, respectively.

Let $A_g = 1$ if ϑ falls within the intervals of FGCI, LS, MOVER, PB or the HPDs, else $A_g = 0$.

The CP and AL for each method are obtained by CP $= (1/M)\sum_{g=1}^{M} A_g$ and AL $= (U - L)/M$, respectively, where U and L are the upper and lower confidence limits, respectively. (end g loop)

The numerical results for the CI performances are presented in terms of CP and AL for various sample cases. For $k = 2$ (Table 2 and Figure 1), FGCI performed well for small-to-moderate sample sizes, as well as for large σ_i^2 and a moderate-to-large sample size. HPD-NGB attained stable and the best CP and AL values for small σ_i^2 and a moderate-to-large sample size. MOVER and PB attained correct CPs but wider ALs than the other methods whereas LS and HPD-JR had lower CPs and narrower ALs. For $k = 5$ (Table 3 and Figure 2), there were only two methods producing better CPs than the other methods in the various situations: MOVER (small δ_i and σ_i^2) and PB (large δ_i and σ_i^2). Moreover, the results were similar for $k = 10$ (Table 4 and Figure 3).

As previously mentioned, our findings show that FGCI works well for small sample case because the FGPQ of σ_i^2 might contain some weak points that affect the FGPQ of μ_i as the sample case increases. For large sample sizes, MOVER was the best method for small σ^2, which is possibly caused by the CI for $\mu_i + \sigma_i^2$. Meanwhile, the next best one was PB, which has the strong point of using a resampling technique to collect information about several populations even when the variance σ^2 is large.

Table 2. Performance measures of 95%CIs for ϑ: 2 sample cases

Scenarios	CP						AL					
	FG	LS	MO	PB	HJ	HN	FG	LS	MO	PB	HJ	HN
$k = 2$												
1	**0.959**	0.897	0.967	0.994	0.916	0.941	**1.556**	1.296	2.005	2.324	1.353	1.436
2	**0.958**	0.857	0.947	0.996	0.924	0.941	**5.169**	3.770	7.287	8.631	4.186	4.335
3	**0.963**	0.821	0.959	0.996	0.919	0.932	**13.088**	8.675	23.312	22.883	9.905	10.220
4	**0.962**	0.886	0.978	0.995	0.917	0.939	**1.487**	1.211	2.181	2.155	1.247	1.386
5	**0.953**	0.832	0.962	0.995	0.913	0.922	**4.875**	3.487	9.881	7.818	3.811	4.066
6	**0.951**	0.793	0.971	0.991	0.901	0.912	**12.311**	7.740	37.615	21.129	8.875	9.378
7	**0.961**	0.829	0.972	0.982	0.920	0.940	**1.511**	1.095	3.968	2.173	1.224	1.406
8	**0.950**	0.778	0.974	0.995	0.900	0.911	**4.821**	3.123	293.620	7.649	3.566	3.916
9	0.939	0.725	0.973	**0.988**	0.866	0.887	13.159	7.067	8.0e4	**23.632**	8.680	9.419
10	**0.960**	0.900	0.965	0.992	0.915	0.941	**1.503**	1.249	1.936	2.225	1.362	1.395
11	**0.961**	0.848	0.941	0.992	0.924	0.940	**5.128**	3.712	6.765	8.667	4.298	4.368
12	**0.965**	0.819	0.952	0.998	0.919	0.931	**12.297**	8.382	20.057	21.597	9.819	9.894
13	**0.960**	0.896	0.977	0.992	0.917	0.942	**1.366**	1.147	1.909	2.004	1.203	1.271
14	**0.961**	0.851	0.964	0.996	0.916	0.931	**4.593**	3.422	7.236	7.458	3.761	3.889
15	**0.949**	0.790	0.958	0.994	0.894	0.905	**11.116**	7.517	22.293	19.310	8.507	8.718
16	**0.963**	0.860	0.972	0.974	0.928	0.943	**1.354**	1.033	2.141	1.928	1.155	1.257
17	**0.952**	0.803	0.976	0.992	0.900	0.917	**4.397**	3.048	10.772	6.889	3.418	3.630
18	0.940	0.737	**0.968**	0.989	0.872	0.889	11.065	6.663	**43.755**	19.011	7.903	8.247
19	**0.961**	0.914	0.966	0.992	0.921	0.946	**1.153**	1.009	1.382	1.696	1.043	1.076
20	**0.965**	0.895	0.946	0.991	0.938	0.949	**3.668**	2.924	4.309	5.981	3.178	3.229
21	**0.962**	0.863	0.952	0.996	0.930	0.940	**8.747**	6.665	11.805	14.651	7.272	7.395
22	**0.958**	0.910	0.978	0.985	0.919	0.944	**1.091**	0.945	1.414	1.555	0.945	1.031
23	**0.965**	0.883	0.969	0.996	0.926	0.937	**3.336**	2.695	4.578	5.204	2.811	2.950
24	**0.961**	0.840	0.972	0.995	0.921	0.928	**7.887**	5.987	13.164	12.757	6.338	6.605
25	**0.969**	0.868	0.980	0.958	0.930	0.953	**1.120**	0.866	1.610	1.503	0.937	1.070
26	**0.954**	0.839	0.970	0.997	0.916	0.926	**3.208**	2.433	6.544	4.735	2.621	2.830
27	**0.946**	0.773	0.970	0.992	0.893	0.903	**7.803**	5.443	26.105	12.382	6.011	6.376
28	**0.958**	0.912	0.972	0.979	0.916	0.947	**1.119**	0.952	1.397	1.615	1.054	1.051
29	**0.956**	0.872	0.921	0.958	0.927	0.943	**3.745**	2.836	4.238	6.098	3.338	3.330
30	**0.961**	0.846	0.937	0.987	0.925	0.936	**8.488**	6.274	10.833	13.991	7.332	7.320
31	**0.962**	0.927	0.985	0.978	0.919	0.949	**0.984**	0.876	1.322	1.433	0.908	0.929
32	**0.960**	0.880	0.958	0.992	0.925	0.940	**3.214**	2.618	4.169	5.150	2.818	2.860
33	**0.958**	0.838	0.960	0.994	0.910	0.925	**7.360**	5.744	10.824	12.105	6.256	6.279
34	**0.963**	0.888	0.977	0.922	0.938	0.954	**0.975**	0.785	1.322	1.352	0.876	0.922
35	**0.958**	0.860	0.971	0.995	0.917	0.929	**2.915**	2.343	4.321	4.424	2.486	2.586
36	**0.951**	0.820	0.973	0.995	0.901	0.916	**6.726**	5.103	11.951	10.823	5.511	5.626
37	**0.957**	0.935	0.960	0.970	0.927	0.948	**0.802**	0.722	0.923	1.168	0.743	0.753
38	**0.955**	0.916	0.926	0.953	0.942	0.948	**2.442**	2.044	2.541	3.935	2.220	2.219
39	**0.957**	0.888	0.939	0.981	0.937	0.945	**5.608**	4.594	6.295	9.049	4.984	4.998
40	**0.961**	0.942	0.975	0.957	0.924	0.954	**0.740**	0.679	0.911	1.062	0.659	0.702
41	**0.961**	0.920	0.960	0.988	0.933	0.950	**2.199**	1.925	2.558	3.401	1.958	2.012
42	**0.955**	0.875	0.960	0.994	0.925	0.931	**4.976**	4.209	6.298	7.813	4.318	4.439
43	**0.967**	0.909	0.980	0.863	0.937	0.960	**0.773**	0.625	0.972	1.012	0.659	0.743
44	**0.960**	0.896	0.970	0.993	0.928	0.939	**2.076**	1.750	2.684	3.013	1.788	1.921
45	**0.952**	0.835	0.970	0.996	0.908	0.914	**4.683**	3.786	7.007	7.008	3.952	4.182

Notes: FG, fiducial generalized confidence interval; MO, method of variance estimates
HJ, HPD-based Jeffreys' rule prior, HPD-JR; HN, HPD-based normal-gamma-beta prior.
Bold denoted as the best-performing method each case.
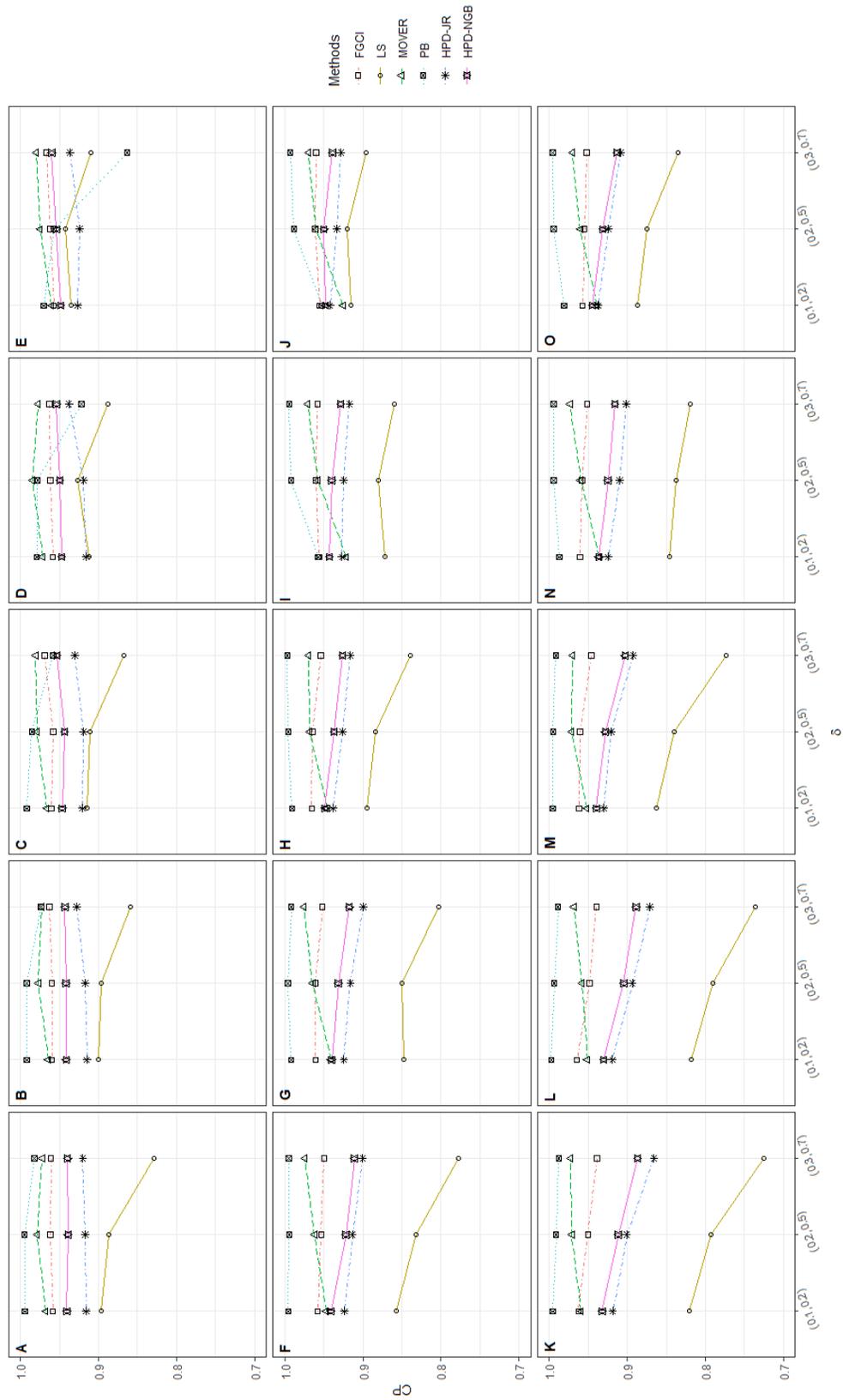
Figure 1. CP performances of 95%CIs for ϑ: 2 sample cases in the following cases (sample sizes, variances): (A) $(30_2, 1, 2)$, (B) $(30, 30, 2, 4)$, (C) $(30, 30, 3, 5)$, (D) $(30, 50, 1, 2)$, (E) $(30, 50, 2, 4)$, (F) $(30, 50, 3, 5)$, (G) $(50_2, 1, 2)$, (H) $(50_2, 2, 4)$, (I) $(50_2, 3, 5)$, (J) $(50, 100, 1, 2)$, (K) $(50, 100, 2, 4)$, (L) $(50, 100, 3, 5)$, (M) $(100_2, 1, 2)$, (N) $(100_2, 2, 4)$, (O) $(100_2, 3, 5)$.

Table 3. Performance measures of 95%CIs for ϑ: 5 sample cases.

Scenarios	CP						AL					
	FG	LS	MO	PB	HJ	HN	FG	LS	MO	PB	HJ	HN
k = 5												
46	0.885	0.790	0.988	**0.989**	0.757	0.846	0.963	0.819	1.794	**1.532**	0.848	0.956
47	0.789	0.627	0.973	**0.996**	0.674	0.715	2.240	1.908	4.982	**3.897**	1.991	2.176
48	0.840	0.613	0.953	**0.997**	0.723	0.746	5.325	4.529	13.769	**12.250**	4.744	4.870
49	0.894	0.800	0.993	**0.978**	0.779	0.864	0.900	0.765	1.825	**1.439**	0.773	0.905
50	0.783	0.623	0.972	**0.998**	0.680	0.711	2.008	1.711	5.203	**3.608**	1.750	1.955
51	0.797	0.580	0.959	**0.996**	0.680	0.701	4.700	4.066	16.626	**11.353**	4.118	4.287
52	0.893	0.735	**0.989**	0.896	0.816	0.853	0.753	0.589	**2.849**	1.433	0.636	0.764
53	0.768	0.517	0.977	**0.997**	0.666	0.676	1.474	1.168	19.967	**3.364**	1.282	1.406
54	0.742	0.467	0.983	**0.996**	0.624	0.629	3.250	2.654	1.5e4	**11.238**	2.817	2.855
55	0.884	0.779	0.988	**0.979**	0.743	0.846	0.940	0.777	1.739	**1.434**	0.857	0.930
56	0.806	0.645	0.973	**0.995**	0.681	0.740	2.204	1.822	4.586	**3.561**	2.045	2.141
57	0.858	0.622	0.949	**0.986**	0.725	0.771	5.620	4.542	12.575	**12.073**	5.122	5.162
58	0.901	0.827	0.995	**0.962**	0.770	0.870	0.845	0.728	1.699	**1.326**	0.771	0.841
59	0.793	0.644	0.978	**0.997**	0.675	0.726	1.904	1.629	4.351	**3.262**	1.750	1.850
60	0.825	0.605	0.952	**0.997**	0.710	0.734	4.753	4.058	12.745	**10.793**	4.373	4.353
61	0.905	0.785	**0.992**	0.822	0.809	0.865	0.685	0.564	**1.632**	1.219	0.620	0.686
62	0.786	0.578	0.969	**0.993**	0.683	0.704	1.368	1.142	4.477	**2.775**	1.260	1.309
63	0.755	0.496	0.963	**0.998**	0.639	0.637	3.177	2.714	18.995	**8.911**	2.884	2.822
64	0.892	0.787	0.991	**0.970**	0.737	0.858	0.928	0.751	1.740	**1.364**	0.872	0.919
65	0.822	0.647	0.975	**0.996**	0.673	0.763	2.168	1.738	4.371	**3.326**	2.047	2.114
66	0.852	0.593	0.943	**0.981**	0.715	0.767	5.710	4.413	12.195	**11.422**	5.267	5.278
67	0.905	0.827	0.996	**0.949**	0.768	0.873	0.816	0.697	1.637	**1.256**	0.770	0.811
68	0.801	0.654	0.979	**0.995**	0.683	0.737	1.839	1.549	4.069	**3.016**	1.753	1.797
69	0.821	0.595	0.947	**0.994**	0.693	0.733	4.806	3.976	12.174	**10.326**	4.431	4.432
70	0.917	0.803	**0.994**	0.775	0.817	0.886	0.650	0.539	**1.499**	1.133	0.616	0.650
71	0.804	0.612	0.973	**0.992**	0.692	0.730	1.310	1.094	3.962	**2.543**	1.236	1.262
72	0.756	0.502	0.958	**0.997**	0.631	0.646	3.158	2.695	16.604	**8.356**	2.888	2.835
73	0.924	0.832	**0.994**	0.942	0.772	0.893	0.822	0.673	**1.505**	1.186	0.856	0.808
74	0.853	0.699	0.985	**0.990**	0.696	0.798	1.971	1.589	3.823	**2.899**	2.000	1.923
75	0.883	0.652	0.952	**0.945**	0.755	0.817	5.330	4.072	9.997	**9.911**	5.224	4.974
76	0.924	0.857	**0.997**	0.913	0.771	0.901	0.723	0.626	**1.418**	1.088	0.746	0.715
77	0.826	0.695	0.986	**0.989**	0.689	0.767	1.670	1.406	3.476	**2.610**	1.692	1.632
78	0.854	0.638	0.955	**0.984**	0.718	0.771	4.456	3.628	9.715	**8.788**	4.311	4.160
79	0.930	0.846	**0.998**	0.683	0.811	0.900	0.581	0.486	**1.253**	0.964	0.586	0.580
80	0.830	0.658	0.981	**0.980**	0.705	0.762	1.215	1.019	3.179	**2.168**	1.225	1.181
81	0.788	0.555	0.967	**0.997**	0.675	0.689	2.992	2.554	11.873	**7.026**	2.927	2.738
82	0.915	0.844	0.993	**0.964**	0.788	0.889	0.769	0.662	1.337	**1.158**	0.692	0.753
83	0.858	0.735	0.982	**0.993**	0.741	0.804	1.882	1.599	3.605	**2.920**	1.698	1.825
84	0.886	0.705	0.969	**0.981**	0.782	0.827	4.650	3.895	8.767	**9.068**	4.208	4.335
85	0.925	0.865	**0.998**	0.939	0.803	0.897	0.707	0.618	**1.315**	1.068	0.618	0.700
86	0.834	0.705	0.987	**0.994**	0.735	0.775	1.683	1.439	3.493	**2.683**	1.482	1.642
87	0.855	0.684	0.968	**0.994**	0.751	0.783	4.027	3.489	8.924	**8.068**	3.613	3.766
88	0.929	0.824	**0.994**	0.677	0.835	0.903	0.611	0.495	**1.322**	0.993	0.515	0.616
89	0.823	0.627	0.981	**0.985**	0.729	0.749	1.284	1.045	3.692	**2.296**	1.121	1.250
90	0.799	0.578	0.972	**0.997**	0.699	0.705	2.875	2.453	13.603	**6.644**	2.519	2.641
91	0.927	0.831	**0.997**	0.906	0.777	0.898	0.753	0.614	**1.389**	1.064	0.703	0.735
92	0.871	0.731	0.988	**0.986**	0.720	0.820	1.821	1.466	3.459	**2.601**	1.721	1.769
93	0.905	0.693	0.957	**0.897**	0.791	0.852	5.015	3.768	8.461	**8.829**	4.621	4.690
94	0.931	0.879	**0.999**	0.873	0.781	0.909	0.651	0.571	**1.279**	0.972	0.608	0.639
95	0.847	0.738	0.991	**0.986**	0.719	0.797	1.541	1.313	3.117	**2.351**	1.447	1.499
96	0.875	0.679	0.966	**0.969**	0.760	0.806	4.125	3.374	8.002	**7.707**	3.808	3.865
97	0.935	0.866	**0.998**	0.541	0.832	0.911	0.529	0.450	**1.097**	0.856	0.493	0.523
98	0.848	0.697	0.986	**0.971**	0.735	0.782	1.126	0.956	2.572	**1.916**	1.060	1.091
99	0.817	0.613	0.963	**0.994**	0.698	0.725	2.784	2.418	7.510	**6.042**	2.565	2.571
100	0.941	0.888	**0.998**	0.863	0.813	0.920	0.557	0.484	**0.954**	0.806	0.510	0.536
101	0.906	0.827	0.995	**0.973**	0.799	0.875	1.413	1.201	2.515	**2.029**	1.288	1.361
102	0.929	0.790	**0.975**	0.861	0.845	0.889	3.639	2.946	5.529	6.174	3.365	3.428
103	0.948	0.923	**1.000**	0.801	0.816	0.931	0.501	0.456	**0.909**	0.741	0.452	0.487
104	0.888	0.816	0.996	**0.978**	0.784	0.853	1.253	1.095	2.373	**1.852**	1.121	1.216
105	0.905	0.775	0.981	**0.953**	0.822	0.859	3.147	2.678	5.326	**5.441**	2.893	2.975
106	0.955	0.907	**0.999**	0.289	0.852	0.943	0.438	0.372	**0.838**	0.668	0.373	0.433
107	0.881	0.761	**0.994**	0.939	0.781	0.833	0.992	0.823	**2.044**	1.536	0.863	0.972
108	0.868	0.722	0.984	**0.987**	0.781	0.805	2.331	2.005	5.072	**4.308**	2.088	2.208

Notes: FG, fiducial generalized confidence interval; MO, method of variance estimates recovery;
HJ, HPD-based Jeffreys' rule prior, HPD-JR; HN, HPD-based normal-gamma-beta prior.
Bold denoted as the best-performing method each case.
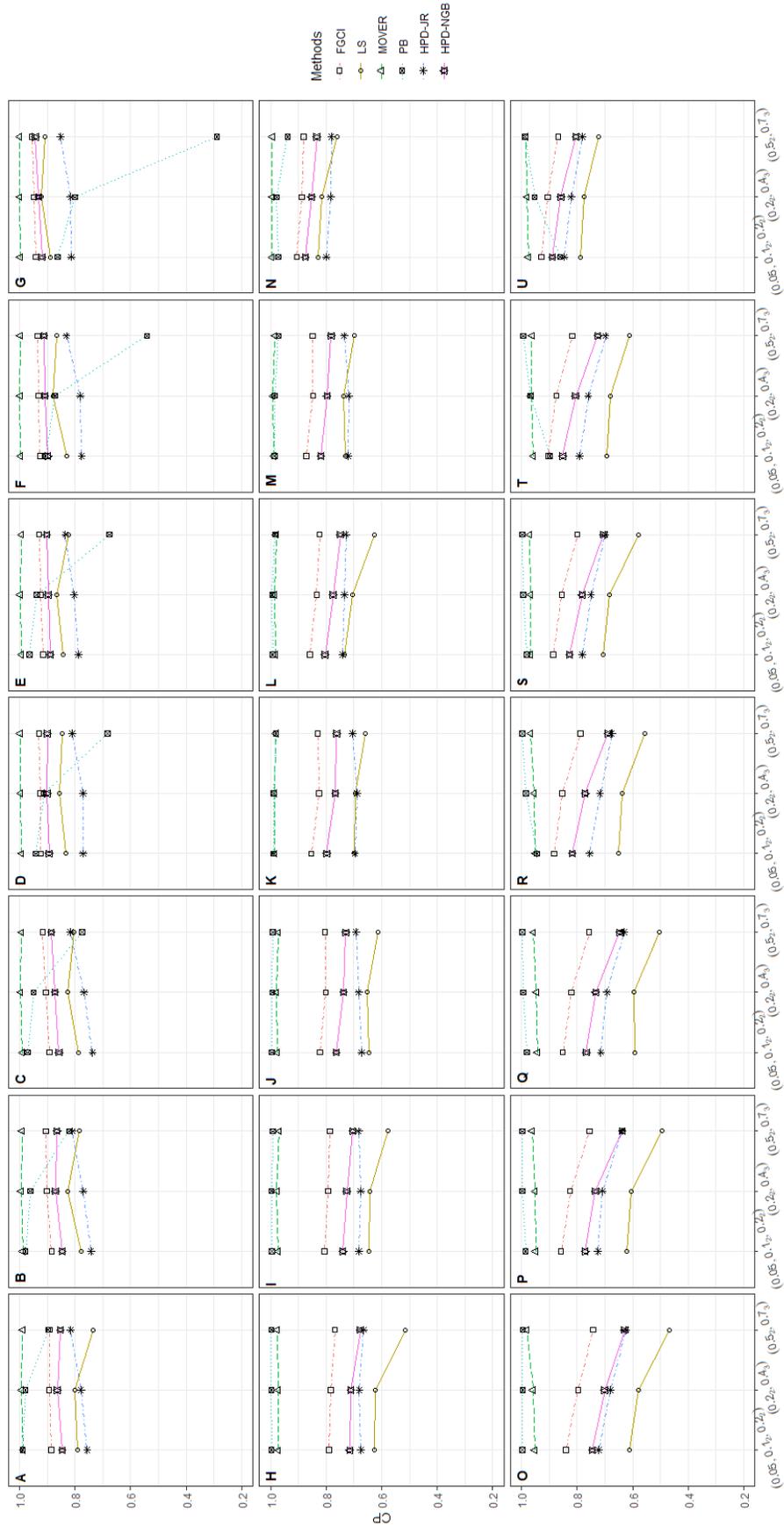
Figure 2. CP performances of 95%CIs for ϑ: 5 sample cases in the following cases (sample sizes, variances): (A) $(30_5, 1_2, 2_3)$, (B) $(30_5, 2_2, 3_3)$, (C) $(30_5, 3_2, 5_3)$, (D) $(30_2, 50_3, 1_2, 2_3)$, (E) $(30_2, 50_3, 2_2, 3_3)$, (F) $(30_2, 50_3, 3_2, 5_3)$, (G) $(30_2, 50_3, 100, 1_2, 2_3)$, (H) $(30_2, 50_3, 100, 2_2, 3_3)$, (I) $(30_2, 50_3, 100, 3_2, 5_3)$, (J) $(30_2, 50_2, 100_2, 1_2, 2_3)$, (K) $(30_2, 50_2, 100_2, 1_2, 2_3)$, (L) $(30_2, 50_2, 100_2, 1_2, 2_3)$, (M) $(50_5, 1_2, 2_3)$, (N) $(50_5, 2_2, 3_3)$, (O) $(50_5, 3_2, 5_3)$, (P) $(50_2, 100_3, 1_2, 2_3)$, (Q) $(50_2, 100_3, 2_2, 3_3)$, (R) $(50_2, 100_3, 3_2, 5_3)$, (S) $(100_5, 1_2, 2_3)$, (T) $(100_5, 2_2, 3_3)$, (U) $(100_5, 3_2, 5_3)$.

Table 4. Performance measures of 95%CIs for ϑ: 10 sample cases.

Scenarios	CP						AL					
	FG	LS	MO	PB	HJ	HN	FG	LS	MO	PB	HJ	HN
k = 10												
109	0.728	0.675	**0.998**	0.927	0.566	0.692	0.612	0.501	**1.554**	0.932	0.545	0.623
110	0.661	0.500	**0.979**	0.891	0.570	0.588	1.644	1.291	**3.867**	3.278	1.500	1.637
111	0.504	0.352	0.950	**0.978**	0.481	0.404	3.159	2.561	8.645	**7.286**	2.996	3.076
112	0.720	0.692	**0.999**	0.904	0.587	0.690	0.557	0.459	**1.519**	0.832	0.483	0.574
113	0.532	0.452	0.976	**0.985**	0.512	0.462	1.393	1.159	3.853	**2.682**	1.260	1.404
114	0.361	0.290	0.955	**0.998**	0.403	0.274	2.556	2.218	8.570	**5.943**	2.411	2.505
115	0.789	0.723	**0.999**	0.808	0.561	0.762	0.554	0.440	**1.416**	0.789	0.546	0.560
116	0.716	0.524	**0.985**	0.578	0.590	0.653	1.635	1.180	**3.478**	2.915	1.559	1.624
117	0.593	0.406	**0.964**	0.872	0.519	0.507	3.289	2.406	**7.754**	6.380	3.189	3.214
118	0.782	0.773	**1.000**	0.780	0.586	0.758	0.477	0.404	**1.317**	0.696	0.474	0.483
119	0.626	0.514	0.988	**0.947**	0.535	0.561	1.337	1.076	3.348	**2.360**	1.284	1.341
120	0.447	0.347	0.965	**0.992**	0.450	0.355	2.570	2.108	7.290	**5.180**	2.506	2.531
121	0.826	0.773	**1.000**	0.736	0.592	0.796	0.488	0.399	**1.266**	0.695	0.444	0.486
122	0.774	0.620	**0.994**	0.438	0.647	0.720	1.460	1.086	**3.072**	2.512	1.328	1.438
123	0.659	0.460	**0.977**	0.798	0.553	0.577	3.002	2.236	**6.597**	5.502	2.775	2.921
124	0.828	0.826	**1.000**	0.708	0.606	0.802	0.426	0.368	**1.187**	0.615	0.387	0.427
125	0.688	0.595	**0.995**	0.912	0.591	0.627	1.205	0.992	**2.912**	2.039	1.094	1.197
126	0.520	0.426	0.979	**0.984**	0.486	0.439	2.390	1.989	6.222	**4.479**	2.224	2.344

Notes: FG, fiducial generalized confidence interval; MO, method of variance estimates recovery;
HJ, HPD-based Jeffreys' rule prior, HPD-JR; HN, HPD-based normal-gamma-beta prior.
Bold denoted as the best-performing method each case.

AN EMPIRICAL APPLICATION

Daily rainfall data obtained from the Thai Meteorological Department (TMD) were divided into the northern, northeastern, central, and eastern regions, while the southern region was a combination of the data from the southeastern and southwestern shores. Due to the differences in the climate patterns and meteorological conditions in the five regions, we focused was on estimating the daily rainfall data in these regions by treating them as separate sets of observations rather than using the average rainfall for the whole of Thailand by pooling them and treating them as a single population. The daily rainfall amounts were recorded on August 5 and 9, 2019, which is in the middle of the rainy season (mid-May to mid-October) when rice farming is conducted in Thailand. Entries with rainfall of less than 0.1 mm were considered as zero records.

Tables 5-6 contain the daily rainfall records for the five regions, while Figures 4-5 show histogram plots of rainfall observations, and Figures 6-7 exhibit normal Q-Q plots of the log-positive rainfall data on August 5 and 9, 2019, respectively. It can be seen that the data for all of the regions contained zero observations. After that, the fitted distribution of the positive observations was checked using the Akaike information criterion (AIC), as reported in Table 7. It can be concluded that the rainfall data in all of the regions on August 5 and 9, 2019 follow a delta-lognormal distribution. All data sets and R code are available in the Supplemental Files. The summary statistics are reported in Table 8. In the approximation of the daily rainfall amounts in the five regions, the estimated common means were 4.4506 and 13.2621 mm/day on August 5 and 9, 2019, respectively. The computed 95%CIs of the common rainfall mean are reported in Table 9. Under the rain criteria issued by the TMD (Department, 2018), it can be interpreted that the daily rainfall in Thailand on August 5, 2019, was light (0.1–10.0 mm), while it was moderate (10.1-35.0 mm) on August 9, 2019. These results confirm the simulation results for $k = 5$ in the previous section.

DISCUSSION

It can be seen that for MOVER and PB developed from the studies of Krishnamoorthy and Oral (2015) and Malekzadeh and Kharrati-Kopaei (2019), respectively, the simulation results are similar to both of these studies provided that the zero observations are omitted. CIs for the common mean have been investigated in both normal and lognormal distributions (Fairweather, 1972; Jordan and Krishnamoorthy, 1996; Krishnamoorthy and Mathew, 2003; Lin and Lee, 2005; Tian and Wu, 2007; Krishnamoorthy and Oral, 2015). However, the common mean of delta-lognormal populations is especially of interest because

Table 5. Daily rainfall data in five Thailand's regions on August 5, 2019.

Northern			Northeastern						Central		Eastern	Southern			
3	0	3	0	0	49.5	0	0	0	2.9	3.2	0	4.1	0	0	2.7
2.6	5	0	40	1.5	10.5	0	0	0	0.2	0	3.2	0	0	0	0
1	23.8	0	3.5	18.5	60.4	4	0	11	0.3	0	10.4	11.5	3.5	0	0
3.6	16	0	0	42	12.7	0	0	0	2.5	4.7	1.1	2.5	13.6	0	0
0	11.5	0	12	9.1	6.8	0	20.3	0	0.4	19.3	0.2	9.7	0	0.2	0
13.2	1.2	0	15	6	69.3	0	0	0	0.4	3.1	4.3	10.4	0	0	0
22.4	10.3	0	0	7.5	36.5	0	2.4	0.3	1.1	2.9	0	9.6	0	0	0
1.4	1.7	0	1.5	0	8.6	0	0	1	0	5.7	0	19	0	0	0
18.3	5.5	0	0.7	6.3	0	0	0	0	1.3	0.9	0	8.3	0	0	0
0	7.3	0	0	0	0	0	0	0	0.1	0	0	0	4.8	0	6.2
15.5	24.3	1.7	3	0.4	0	0	0	0	2.9	0	0.2	0	0	0	0
0	27.2	2.3	0	0	3.8	0	0	0	0	2.6	0.1	0	0	0	0
0	12.6	0.5	0	0	0	0	3.2	0	1	17	62.8	0	0	0	6.1
0	22.7	3.9	0	0	0	0	0	0	4.7	0	36.7	17.8	0	0	0
9.8	0	6.9	29.4	1.8	0	0	0	0	0.5	3.5	15.6	12.3	0	0	0
24.3	2.6	2.2	48	0	0	0	0	0	5	0	50	2.5	0	0	0
24.6	0	3.2	0	0	0	6	0	0	2.5	0	35.5	0	0	0	0.3
8.8	3.2	5.3	70.8	14.3	0	0	0	0	0	0	35	0.9	0	0	0
0	2.6	11	3.5	0	0	0	0	0	0	5.1	5.9	0	0	0	0
19.8	2	0.6	14.2	0	0	0	4.8	0	0	60.4	0	2.6	0	0	0
5	8	0	7	0	0	2.3	0	0	0	6.9	0	0	0	0	0
12.3	1.9	1	0	0	21.5	0	0	0	6.6	3	3	0	0	0	0
8.1	0.8	2.4	0	0	2.5	1	0	0	0	15.1	60.4	2	0	0	0
4.8	2.2	13.2	0	0	0	0	0	0	9.5	6	60	0	0	0	0
5.8	6.5	0.4	0	0	13	0	0		5.1	13.4	76	0	0	0	0
17	0	0	10.8	0	26.2	0	0		12.5	6.2	79.7	0	0	0	0
25.1	2.2	1.3	0	10.1	2.2	4.6	5.4		0		65.7	3.5	0	0	
8.3	0	10	6.3	0	3	0	0		0		108	0	0	36.1	
22.9	4.3	2.5	0	4.8	10.5	10	0		3.2		10.5	0	0	41.8	
26.9	0.2	4.6	4	0	0	0	12		0			0	0	30	
0	0	0	19.3	0	0	9.5	0		2.2			0	0	0	

Source: Thai Meteorological Department
https://www.tmd.go.th/services/weekly_report.php

it can be used to fit the data from real-world situations such as investigating medical costs (Zou et al., 2009; Tierney et al., 2003; Tian, 2005), analyzing airborne contaminants (Owen and DeRouen, 1980; Tian, 2005) and measuring fish abundance (Fletcher, 2008; Wu and Hsieh, 2014). Furthermore, it is possible that some extreme rainfall data also fulfill the assumptions of a delta-lognormal distribution. Note that such natural disasters as floods and landslides have been caused by the extreme rainfall events, as evidenced in many country around the world: Europe (e.g. Northern England, Southern Scotland and Ireland (Otto and Oldenborgh, 2017)), Asia (e.g. Japan (Oldenborgh, 2018)) and North America (e.g. Southeast Texas (Oldenborgh et al., 2019)). Our findings show that some of the methods studied had CPs that were too low or too high for large sample cases, a shortcoming that should be addressed in future work.

CONCLUSIONS

The objective of this study was to propose CIs for the common mean of several delta-lognormal distributions using FGCI, LS, MOVER, PB, HPD-JR, and HPD-NGB. The CP and AL as performance measures of the methods were assessed via Monte Carlo simulation. The findings confirm that for small sample case ($k = 2$), FGCI and HPD-NGB are the recommended methods in different situations: FGCI (a small-to-moderate sample size and a large σ_i^2 with a moderate-to-large sample size) and HPD-NGB (small σ_i^2 with a moderate-to-large sample size). For large sample cases ($k = 5, 10$), MOVER (small δ_i and σ_i^2) and PB (large δ_i and σ_i^2) performed the best.

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Table 6. Daily rainfall data in five Thailand's regions on August 9, 2019.

Northern		Northeastern							Central		Eastern	Southern			
9.5	0	25.3	20	6.6	8.4	0	67	0	39.6	0	0	27.9	4.1	0.4	114.6
4.9	10	25.5	14.5	16.9	0.8	2.9	65.4	0	25	0	0	0	9	3.8	0
0	21.6	24	3	10	20.2	0	21	0	0	0	26.5	3.4	27.3	0.6	0
4.7	15	8	28	48.2	0	14.3	6.4	7.2	0	0	36.4	0	6.5	0	0
0	15.5	0	27	6.5	0.5	0	0	3.5	29.7	0.1	0	0.8	3.5	10.8	0
63.2	14	20	50	4.8	5.3	6	52	0	0	0.3	4.5	37.9	0	5	18.2
9.6	8.5	0	24	25	16.7	0	45	40.5	3.1	0.5	0	32.4	0	12.2	40.4
10.7	11.5	0	30	0	45.2	28	41.4	25.8	8.2	31.5	0.5	33.8	0	3.6	0
13	17.4	0	22	0	0	0	14.3	30.4	3.2	8.2	0.7	15.8	0	0	0
0	15.6	0	16	3.2	0.6	0	45	0	7.1	0	12.3	0	3.6	8.8	10.8
0	31.6	33.8	0	44	0	0	27	0	0	0	0.5	0	3	0	0
0	20.6	33.7	0	0	3.1	27.6	0.2	0	3.2	0	1.9	0	1	0	0
0	31.1	15.1	0	9.3	33.3	33	30	0	4.2	0	66.4	0	3.7	6.2	35
0	16.3	18.5	0	0	6	0	0	0	5.7	0	93.6	11.5	15.6	0	0
2.8	0	44.8	39.7	20	0	0	0	8.3	30	0	68.7	1.7	11.2	3.8	33.5
11.3	33.1	37.5	9.3	0	13.2	0	0	0	4	0	40	1.2	24	0	57
0.6	29.2	0	0	4.8	0	0	0	0	0	0	65	21.2	0	0	10.5
36.1	11.2	47	2.1	0	21	0	0	0	0	0	63.7	0	0	0	0
0	14.4	20	0	0	0	0	1	36.1	0	0	9.2	30	10.2	0.2	0
2.6	60	30.8	46.7	0	8.4	15	0	0	0	1.2	0	5.1	0	0	0
5	42.3	30	10.5	0	0	0	0	12.5	0	0	0	2.5	0	0	30.8
13.4	9.5	1	0	56.5	0	0	2.5	0	14.7	0.1	11	2.4	0	0.4	10.7
12.3	34.5	1.2	41	39.2	0.5	0	0	0	0	1	69.6	5	0	0	0
25.8	36.5	56.3	10.3	0	4.5	25.7	9.5	0	0	3	89.6	1.7	0	0	15.9
30.2	9.7	0	1.2	6.4	16.2	41.4	0		0	0.5	160	0	0	2.2	0
16.4	0	6	23.9	5.3	0	41.6	0		0	1.6	34.3	0	0	0	0
6	0	0	22.2	0	3.5	53.8	0		0		0	2.1	0	0.6	
33.1	7.6	5.3	24.1	9.8	20	48.5	0		0		25	0	0	76.6	
16.4	9.6	7.2	38	0	0	78.5	2.1		0		19.5	10.5	7	121.6	
19.8	9.3	24.6	9	9.7	0	12.7	0		0			15.3	10.6	60	
0	0	30	9.2	4.5	1.2	80.9	0		0			0	3.6	0	

Source: Thai Meteorological Department
URL: https://www.tmd.go.th/services/weekly_report.php

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Table 7. AIC results of daily rainfall records in five Thailand's regions.

Regions	AIC				
	Cauchy	Logistic	Lognormal	Normal	T-distribution
On August 5, 2019					
Northern	373.1958	357.3122	**336.8724**	353.7757	354.3055
Northeastern	600.9473	642.1779	**543.9619**	667.2334	664.6152
Central	240.0227	266.4162	**220.8503**	293.9151	283.2302
Eastern	229.8995	220.2523	**202.8394**	218.7240	219.1471
Southern	194.9368	197.5586	**178.5587**	201.1654	200.1388
On August 9, 2019					
Northern	389.6257	387.3072	**375.7994**	391.1802	390.2479
Northeastern	1123.7491	1080.8694	**1052.8953**	1080.1467	1079.9365
Central	178.8516	189.5353	**155.0261**	190.6855	190.5103
Eastern	233.5236	227.1725	**215.9306**	228.0501	227.4559
Southern	541.0477	569.2615	**487.4667**	592.2242	588.2377

Table 8. The summary statistics.

Regions	Estimated parameters				
	n_i	$\hat{\mu}_i$	$\hat{\sigma}_i^2$	$\hat{\delta}_i$	$\hat{\vartheta}_i$
August 5, 2020					
Northern	62	1.866	1.277	0.210	9.472
Northeastern	210	1.734	1.578	0.619	4.668
Central	57	1.085	1.784	0.316	4.741
Eastern	29	2.366	4.545	0.241	59.391
Southern	119	1.684	1.730	0.782	2.639
August 9, 2020					
Northern	62	2.621	0.732	0.226	15.187
Northeastern	210	2.577	1.502	0.405	16.429
Central	57	1.190	3.054	0.579	5.542
Eastern	29	2.860	3.070	0.241	52.813
Southern	119	2.007	2.051	0.462	10.811

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Table 9. 95%CIs of common rainfall mean in five Thailand's regions.

Methods	95%CIs for ϑ		Lengths
	Lower	Upper	
On August 5, 2020			
FGCI	2.5545	6.3342	3.7798
LS	3.2166	5.6846	2.4681
MOVER	2.7216	9.0296	6.3080
PB	5.8876	11.4965	5.6089
HPD-JR	3.5216	7.8533	4.3317
HPD-NGB	2.4969	6.0904	3.5935
On August 9, 2020			
FGCI	7.1127	16.8809	9.7682
LS	10.4880	16.0363	5.5483
MOVER	7.5814	23.3171	15.7357
PB	14.5229	23.5821	9.0591
HPD-JR	12.8404	20.4349	7.5945
HPD-NGB	7.2928	17.1265	9.8337

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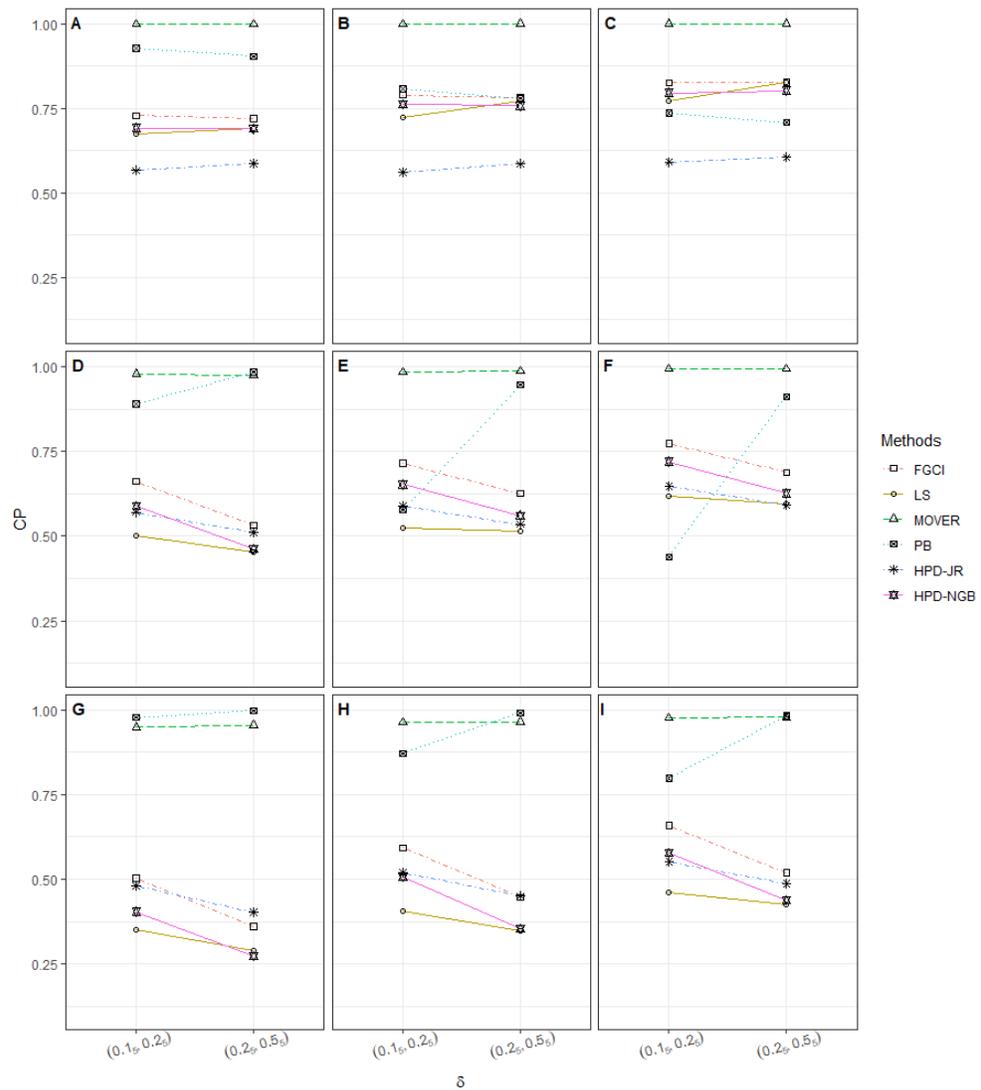
Figure 3. CP performances of 95%CIs for ϑ: 10 sample cases in the following cases (sample sizes, variances): (A) $(30_5, 50_5, 1_2, 2_5)$, (B) $(30_5, 50_5, 2_5, 4_5)$, (C) $(30_5, 50_5, 3_5, 5_5)$, (D) $(30_3, 50_3, 100_4, 1_2, 2_5)$, (E) $(30_3, 50_3, 100_4, 2_5, 4_5)$, (F) $(30_3, 50_3, 100_4, 3_5, 5_5)$, (G) $(50_5, 100_5, 1_5, 2_5)$, (H) $(50_5, 100_5, 2_5, 4_5)$, (I) $(50_5, 100_5, 3_2, 5_3)$.
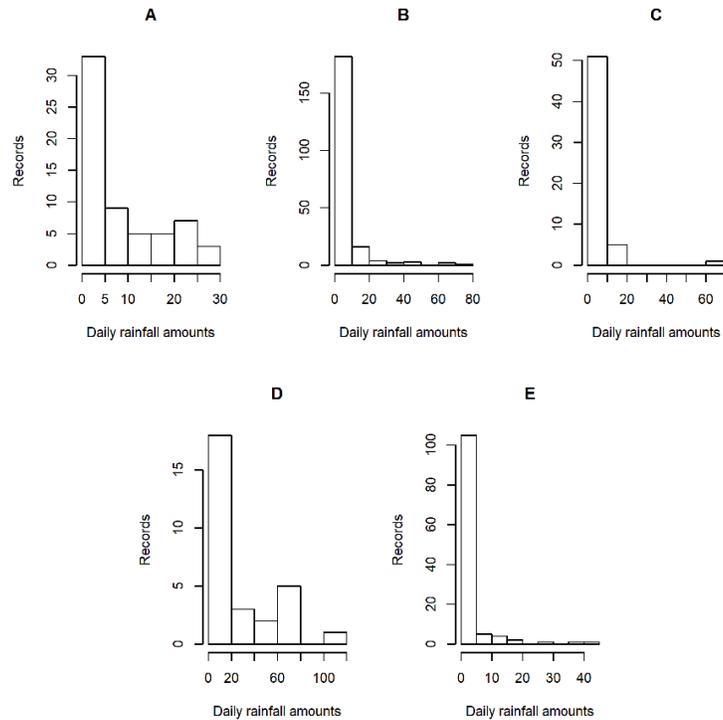
Figure 4. Histogram plots of daily rainfall data in five Thailand's regions on August 5, 2019: (A) Northern (B) Northeastern (C) Central (D) Eastern (E) Southern.
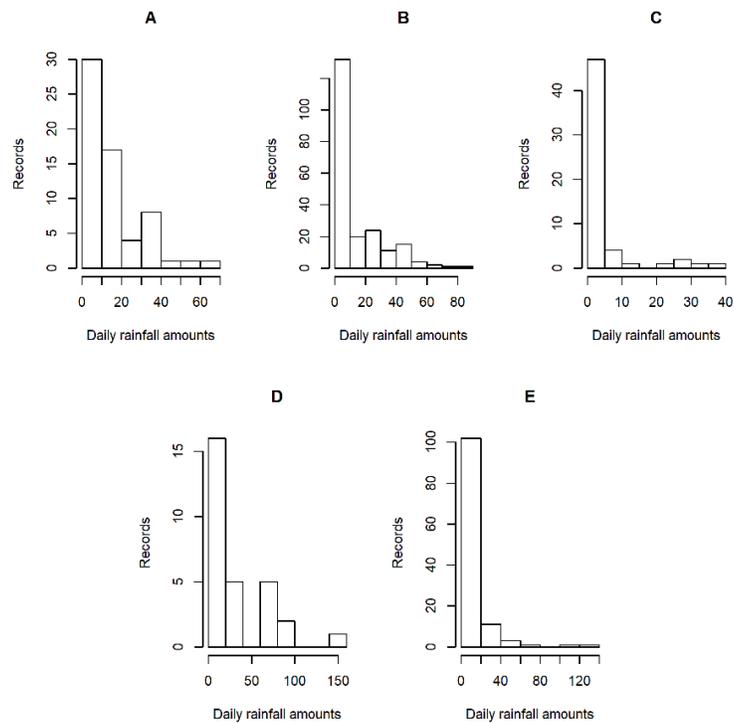
Figure 5. Histogram plots of daily rainfall data in five Thailand's regions on August 9, 2019: (A) Northern (B) Northeastern (C) Central (D) Eastern (E) Southern.
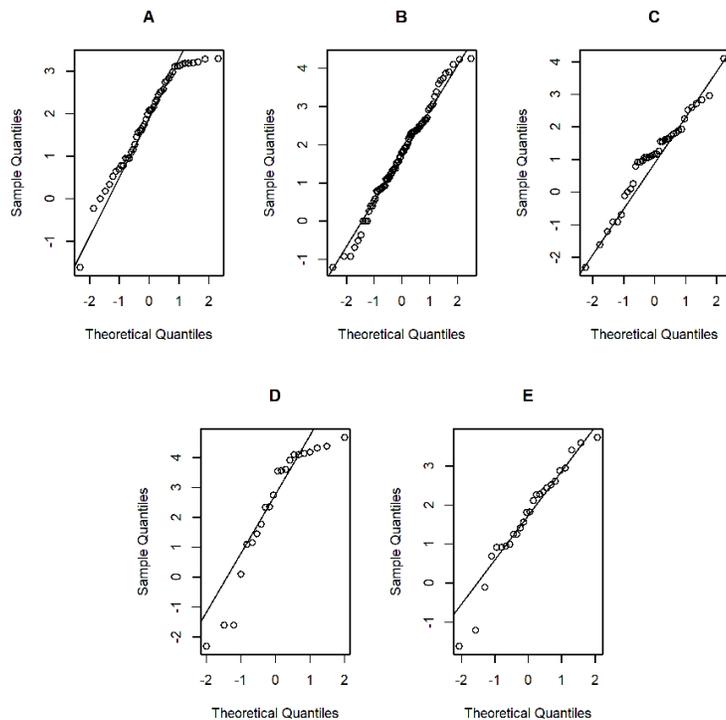
Figure 6. Normal Q-Q plots of log-positive daily rainfall data in five Thailand's regions on August 5, 2019: (A) Northern (B) Northeastern (C) Central (D) Eastern (E) Southern.
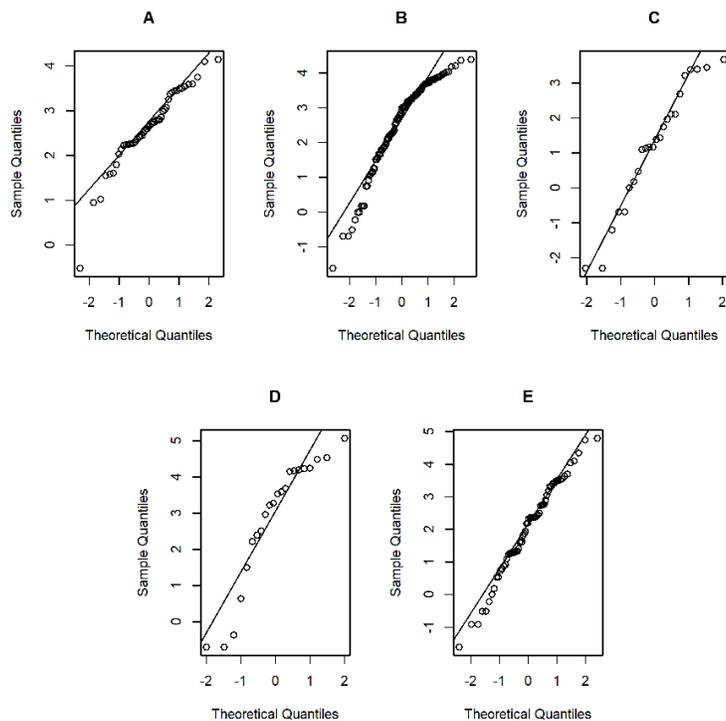
Figure 7. Normal Q-Q plots of log-positive daily rainfall data in five Thailand's regions on August 9, 2019: (A) Northern (B) Northeastern (C) Central (D) Eastern (E) Southern.