Estimating the average daily rainfall in Thailand using confidence intervals for the common mean of several delta-lognormal distributions

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ABSTRACT

The daily average natural rainfall amounts in the five regions of Thailand can be estimated using the confidence intervals for the common mean of several delta-lognormal distributions based on the fiducial generalized confidence interval (FGCI), large sample (LS), method of variance estimates recovery (MOVER), parametric bootstrap (PB), and highest posterior density intervals based on Jeffreys' rule (HPD-JR) and normal-gamma-beta (HPD-NGB) priors. Monte Carlo simulation was conducted to assess the performance in terms of the coverage probability and average length of the proposed methods. The numerical results indicate that MOVER and PB provided better performances than the other methods in a variety of situations, even when the sample case was large. The efficacies of the proposed methods were illustrated by applying them to real rainfall datasets from the five regions of Thailand.

INTRODUCTION

Approximately 82.2% of Thailand's cultivated land area depends on natural rainfall (Supasod, 2006) which indicates that it is an important factor in Thai agriculture. However, it is natural phenomenon with a significant level of uncertainty. The annual rainfall mean is approximately 1572.5 mm in Thailand (Thai Meteorological Department, 2014). However, it is necessary to assess how rainfall varies in each region on a daily basis. Due to the climate pattern and meteorological conditions, Thailand is commonly separated into five regions: northern, northeastern, central, eastern and southern. The rainfall in each region is different because it widely varies due to both location and season. Importantly, Thailand's rainfall data include many zeros with probability $\delta > 0$ and positively right-skewed data having a lognormal distribution for the remainder of the probability. Thus, applying a delta-lognormal distribution (Aitchison, 1955) is appropriate.

The mean is a measure of center (Casella and Berger, 2002) that can be used in statistical inference. Furthermore, functions of the mean such as the ratio or difference between two means are of interest. These parameters have been applied in many research areas, such as medicine, fish stocks, pharmaceutics, and climatology. For example, they have been used for hypothesis testing of the effect of race on the average medical costs between African American and Caucasian patients with type I diabetes (Zhou et al., 1997), to estimate the mean charges for diagnostic tests on patients with unstable chronic medical conditions (Zhou and Tu, 2000; Tian, 2005; Tian and Wu, 2007; Li et al., 2013), to estimate the maximum alcohol concentration in men in an alcohol interaction study (Tian and Wu, 2007; Krishnamoorthy and Oral, 2015), to estimate the mean of red cod density around New Zealand as an indication of fish abundance (Fletcher, 2008; Wu and Hsieh, 2014), and to estimate the mean of the monthly rainfall totals to compare rainfall in Bloemfontein and Kimberley in South African (Harvey and van der Merwe, 2012).

In practice, the mean has been widely used in many fields, as mentioned before. If independent samples are recorded from several situations, then the common mean is of interest when studying more than one population. Many researchers have investigated methods for constructing confidence interval (CIs) for the common mean of several distributions. For example, Fairweather (1972) proposed a linear combination of Student's t to construct CIs for the common mean of several normal distributions. Jordan and Krishnamoorthy (1996) solved the problem of CIs for the common mean under unknown and unequal variances based on Student's t and independent F variables from several normal populations. Krishnamoorthy and Mathew (2003) presented the generalized CI (GCI) and compared it with the CIs constructed by Fairweather (1972), and Jordan and Krishnamoorthy (1996). Later, Lin and Lee (2005) developed the GCI for the common mean of several normal populations. Tian and Wu (2007) provided CIs for the common mean of several lognormal populations using the generalized variable approach, which was shown to be consistently better than the large sample (LS) approach. Lin and Wang (2013) studied the modification of the quadratic method to make inference via hypothesis testing and interval estimation for several lognormal means. Krishnamoorthy and Oral (2015) proposed the method of variance estimates recovery (MOVER) approach for the common mean of lognormal distributions.

As mentioned earlier, most researchers have developed CIs for the common mean of several normal and lognormal distributions. However, there has not yet been an investigation of statistical inference using the common mean of several delta-lognormal distributions. The daily rainfall records from the five regions Thailand satisfies the assumptions for a delta-lognormal distribution, and there is an important need to estimate the daily rainfall trend in these regions.

Herein, CIs for the common mean of several delta-lognormal models based on the fiducial GCI (FGCI), LS, MOVER, parametric bootstrap (PB), and highest posterior density intervals based on Jeffreys' rule (HPD-JR) and normal-gamma-beta (HPD-NGB) priors are proposed. The outline of this article is as follows. The ideas behind the proposed methods are detailed in Section 2. Numerical computations are reported in Section 3. In Section 4, the daily natural rainfall records of the five regions in Thailand are used to illustrate the efficacy of the methods. The paper is closed with a discussion and conclusions.

METHODS

Let $W_{ij} = (W_{i1}, W_{i2}, ..., W_{in_i})$ be random samples drawn from delta-lognormal distribution; $i = 1, 2, ..., k$ and $j = 1, 2, .., n_i$. There are three parameters: μ_i, σ_i^2 and δ_i in this distribution. The distribution of W_{ij} is given by

$$H(w_{ij}; \mu_i, \sigma_i^2, \delta_i) = \begin{cases} \delta_i & ; w_{ij} = 0 \\ \delta_i + (1-\delta_i) G(w_{ij}; \mu_i, \sigma_i^2) & ; w_{ij} > 0 \end{cases} \quad (1)$$

where $G(w_{ij}; \mu_i, \sigma_i^2)$ is a lognormal distribution function, denoted as $LN(\mu_i, \sigma_i^2)$ so that $\ln W_{ij} \sim N(\mu_i, \sigma_i^2)$. The number of zeros has a binomial distribution $n_{i(0)} = \#\left\{j : w_{ij} = 0\right\} \sim B(n_i, \delta_i)$ where δ_i be a probability of obtaining zero. The population mean of W_{ij} is

$$\vartheta_i = (1-\delta_i)\exp(\mu_i + \frac{\sigma_i^2}{2}) \quad (2)$$

The unbiased estimates of μ_i, σ_i^2 and δ_i are $\hat{\mu}_i = n_{i(1)}^{-1} \sum_{j:w_{ij}>0} \ln W_{ij}$ and $\hat{\sigma}_i^2 = (n_{i(1)} - 1)^{-1} \sum_{j:w_{ij}>0} \left[\ln W_{ij} - \hat{\mu}_i\right]^2$ and $\hat{\delta}_i = n_{i(0)}/n_i$, respectively, where $n_i = n_{i(0)} + n_{i(1)}$; $n_{i(1)} = \#\left\{j : w_{ij} > 0\right\}$. According to Graybill and Deal (1959), and Tian and Wu (2007), the common delta-lognormal mean is defined as

$$\vartheta = \frac{\sum_{i=1}^k w_i \vartheta_i}{\sum_{i=1}^k w_i} \quad (3)$$

where $w_i = 1/var(\hat{\vartheta}_i)$ is the weight based on the i^{th} sample which is inversely proportional to its sample variance. Let $\hat{\vartheta}_i^* = (1-\hat{\delta}_i)\exp(\hat{\mu}_i + \frac{\hat{\sigma}_i^2}{2})$ be the estimates of ϑ_i. According to Longford (2009), the expected value of $\hat{\vartheta}_i^*$is

$$E[\hat{\vartheta}_i^*] = \left[1 - E(\hat{\delta}_i)\right] E\left[\exp\left\{\hat{\mu}_i + \frac{\hat{\sigma}_i^2}{2}\right\}\right] \tag{4}$$

$$= (1-\delta_i)\exp\left(\mu_i + \frac{\sigma_i^2}{n_{i(1)}}\right)\left(\frac{l_i}{l_i - \sigma_i^2}\right)^{l_i/2} \tag{5}$$

where $\hat{\delta}_i \sim N(\delta_i, \frac{\delta_i(1-\delta_i)}{n_i})$ as $n_i \to \infty$, $E[\exp(\hat{\mu}_i)] = \exp\left(\mu_i + \frac{\sigma_i^2}{2n_{i(1)}}\right)$ and $E[\exp(c_i Y_i)] = (1-2c_i)^{-l/2}$; $Y_i = l_i \frac{\hat{\sigma}_i^2}{\sigma_i^2} \sim \chi^2_{l_i}$ and $c_i = \frac{\sigma_i^2}{2l_i}$, $\hat{\sigma}_i^2 = (n_{i(1)}-1)^{-1}\sum_{j=1}^{n_{i(1)}}\left[\ln(W_{ij}) - \hat{\mu}_i\right]^2$. For $\frac{l_i - \sigma_i^2}{l_i} = \exp\left[\frac{-2\sigma_i^2}{l_i}(\frac{1}{2} - \frac{1}{2n_{i(1)}})\right]$, we obtain that

$$\begin{aligned} E[\hat{\vartheta}_i^*] &= (1-\delta_i)\exp\left(\mu_i + \frac{\sigma_i^2}{2n_{i(1)}}\right)\left\{\exp\left[\frac{-2\sigma_i^2}{l_i}(\frac{1}{2} - \frac{1}{2n_{i(1)}})\right]\right\}^{-l_i/2} \\ &= (1-\delta_i)\exp\left(\mu_i + \frac{\sigma_i^2}{2}\right) \end{aligned} \tag{6}$$

which has a limitation to obtain the unbiased estimate $\hat{\vartheta}_i$. According to Aitchison and Brown (1963), the Aitchison estimate of ϑ_i is expressed as

$$\hat{\vartheta}_i^{(Ait)} = \begin{cases} 0 & ;n_{i(1)} = 0 \\ w_{i1}/n_i & ;n_{i(1)} = 1 \\ (1-\hat{\delta}_i)\exp(\hat{\mu}_i)\psi_{n_{i(1)}}(\frac{\hat{\sigma}_i^2}{2}) & ;n_{i(1)} > 1 \end{cases} \tag{7}$$

where $\psi_a(b)$ is a Bessel function is defined as

$$\psi_a(b) = 1 + \frac{(a-1)b}{a} + \frac{(a-1)^3}{a^2 2!}\frac{b^2}{a+1} + \frac{(a-1)^5}{a^3 3!}\frac{b^3}{(a+1)(a+3)} + \ldots \tag{8}$$

For investigating the unbiased estimate $\hat{\vartheta}_i^{(Ait)}$, the expected value is

$$\begin{aligned} E\left[\hat{\vartheta}_i^{(Ait)}\right] &= \sum_{j=1}^{n_i} P(n_{i(1)} = j)E\left[\hat{\vartheta}_i | n_{i(1)} = j\right] \\ &= 0 + P(n_{i(1)} = 1)E\left[w_{i1}/n_i\right] + \sum_{j=2}^{n_i} P(n_{i(1)} = j)E\left[\hat{\vartheta}_i | n_{i(1)} = j\right] \\ &= P(n_{i(1)} = 1)\frac{\exp(\mu_i + \frac{\sigma_i^2}{2})}{n_i} + \sum_{j=2}^{n_i} P(n_{i(1)} = j)E\left[\frac{n_{i(1)}}{n_i}\exp(\mu_i + \frac{\sigma_i^2}{2}) | n_{i(1)} = j\right] \\ &= \sum_{j=0}^{n_i} P(n_{i(1)} = j)E\left[\frac{n_{i(1)}}{n_i}\exp(\mu_i + \frac{\sigma_i^2}{2}) | n_{i(1)} = j\right] \\ &= E\left[\frac{n_{i(1)}}{n_i}\exp(\mu_i + \frac{\sigma_i^2}{2})\right] \\ &= (1-\delta_i)\exp(\mu_i + \frac{\sigma_i^2}{2}) \end{aligned}$$

According to Shimizu and Iwase (1981), the uniformly minimum variance unbiased (UMVU) estimate of ϑ_i is

$$\hat{\vartheta}_i^{(Shi)} = \begin{cases} 0 & ;n_{i(1)} < 1 \\ \frac{n_{i(1)}}{n_i}\exp(\hat{\mu}_i)\,{}_0F_1\left(\frac{n_{i(1)}-1}{2}; \frac{n_{i(1)}-1}{4n_{i(1)}}S_i^2\right) & ;n_{i(1)} \geq 1 \end{cases} \tag{9}$$

where $S_i^2 = \sum_{j=1}^{n_{i(1)}} \left[\ln(W_{ij}) - \hat{\mu}_i\right]^2$ and ${}_0F_1(a;z) = \sum_{m=0}^{\infty} \frac{z^m}{(a)_m m!}$;

$$(a)_m = \begin{cases} 1 & ; m = 0 \\ a(a+1)\ldots(a+m-1) & ; m \geq 1 \end{cases} \tag{10}$$

From Kunio (1983), $E\left[{}_0F_1\left(\frac{n_{i(1)}-1}{2}; \frac{a}{2}S_i^2\right)\right] = \exp(a\sigma^2)$ is obtained, then

$$\begin{aligned} E\left[\hat{\vartheta}_i^{(Shi)}\right] &= E\left[\frac{n_{i(1)}}{n}\exp(\hat{\mu}_i)_0F_1\left(\frac{n_{i(1)}-1}{2}, \frac{n_{i(1)}-1}{4n_{i(1)}}S_i^2\right)\right] \\ &= \frac{n_i(1-\delta_i)}{n_i}\exp\left[\mu_i + \frac{\sigma_i^2}{2n_{i(1)}}\right]\exp\left[\frac{n_{i(1)}-1}{2n_{i(1)}}\sigma_i^2\right] \\ &= (1-\delta_i)\exp\left(\mu_i + \frac{\sigma_i^2}{2}\right) \end{aligned} \tag{11}$$

where $E(n_{i(1)}) = n_i(1-\delta_i)$. The asymptotically variance of $\hat{\vartheta}_i^{(Shi)}$ is given by

$$\begin{aligned} Var\left[\hat{\vartheta}_i^{(Shi)}\right] &= \exp(2\mu_i + \sigma_i^2)\Bigg[\frac{1}{n_i^2}\sum_{j=1}^{n_i}\binom{n_i}{j}(1-\delta_i)^j\delta^{n_i-j}j^2\exp\left(\frac{\sigma_i^2}{j}\right) \\ &\quad {}_0F_1\left(\frac{j-1}{2}; \frac{(j-1)^2}{4j^2}\sigma_i^4\right) - (1-\delta_i)^2\Bigg] \\ &= \frac{\exp(2\mu_i + \sigma_i^2)}{n_i}\left[\delta_i(1-\delta_i) + \frac{1}{2}(1-\delta_i)(2\sigma_i^2 + \sigma_i^4)\right] + O(n^{-2}) \end{aligned} \tag{12}$$

Actually, $\psi_{n_{i(1)}}(\frac{\hat{\sigma}_i^2}{2}) = {}_0F_1\left(\frac{n_{i(1)}-1}{2}; \frac{n_{i(1)}-1}{4n_{i(1)}}S_i^2\right)$ such that $\hat{\vartheta}_i^{(Shi)}$ and $\hat{\vartheta}_i^{(Ait)}$ are the unbiased estimates of ϑ_i under the different ideas, although their variances are the same as $Var\left[\hat{\vartheta}_i^{(Shi)}\right] = Var\left[\hat{\vartheta}_i^{(Ait)}\right]$. Using $\hat{\mu}_i, \hat{\sigma}_i^2$, $\hat{\delta}_i$ from the samples, the estimated delta-lognormal mean $\hat{\vartheta}_i^{(Ait)}$ and variance of $\hat{\vartheta}_i^{(Ait)}$ are obtained. The following methods are detailed to construct the CIs for the common delta-lognormal mean.

Fiducial Generalized Confidence Interval

The fiducial inference was introduced by Fisher (1930). The Fisher's fiducial argument was developed to discover a generalized fiducial recipe extended to the application of fiducial ideas (Hannig, 2009). The concept of fiducial interval has been advanced as the ideas of GPQ such that it is directly used to apply for generalized inference. Later, Hannig et al. (2006) have argued that a subclass of GPQs is called fiducial generalized pivotal quantity (FGPQ), furthermore it provides a framework to show the connection between a distribution and a parameter. Recall that $\hat{\mu}_i \sim N(\mu_i, \sigma_i^2/n_{i(1)})$ and $(n_{i(1)}-1)\hat{\sigma}_i^2/\sigma_i^2 \sim \chi^2_{n_{i(1)}-1}$ are the independent random variables. The structure functions of $\hat{\mu}_i$ and $\hat{\sigma}_i^2$ are

$$\hat{\mu}_i = \mu_i + V_i\sqrt{\frac{\sigma_i^2}{n_{i(1)}}}, \qquad \hat{\sigma}_i^2 = \frac{\sigma_i^2 U_i}{n_{i(1)}-1} \tag{13}$$

which are the function of V_i and U_i, respectively; $V_i \sim N(0,1)$ and $U_i \sim \chi^2_{n_{i(1)}-1}$. Given observed values, the estimates $\hat{\mu}_i$ and $\hat{\sigma}_i^2$ are obtained. The unique solution of $\left(\hat{\mu}_i, \hat{\sigma}_i^2\right) = \left(\mu_i + V_i\sqrt{\frac{\sigma_i^2}{n_{i(1)}}}, \frac{\sigma_i^2 U_i}{n_{i(1)}-1}\right)$ is

$$\mu_i = \hat{\mu}_i - V_i\frac{\hat{\sigma}_i}{\sqrt{n_{i(1)}}}\sqrt{\frac{n_{i(1)}-1}{U_i}}, \qquad \sigma_i^2 = \frac{(n_{i(1)}-1)\hat{\sigma}_i^2}{U_i} \tag{14}$$

The FGPQs of μ_i and σ_i^2 are

$$G_{\mu_i} = \hat{\mu}_i - V_i^* \frac{\hat{\sigma}_i}{\sqrt{n_{i(1)}}} \sqrt{\frac{n_{i(1)} - 1}{U_i^*}} \tag{15}$$

$$G_{\sigma_i^2} = \frac{(n_{i(1)} - 1)\hat{\sigma}_i^2}{U_i^*} \tag{16}$$

where V_i^* and U_i^* are a independent copy of V_i and U_i, respectively. Hasan and Krishnamoorthy (2018) developed the FGPQ of δ_i using beta distribution as $G_{\delta_i'} \sim Beta(\alpha_i, \beta_i)$; $\alpha_i = n_{i(1)} + 0.5$ and $\beta_i = n_{i(0)} + 0.5$. The FGPQ of ϑ based on k individual samples is

$$G_{\vartheta} = \frac{\sum_{i=1}^{k} G_{w_i} G_{\vartheta_i}}{\sum_{i=1}^{k} G_{w_i}} \tag{17}$$

where $G_{\vartheta_i} = G_{\delta_i'} \exp(G_{\mu_i} + G_{\sigma_i^2}/2)$ and $G_{w_i} = 1/G_{Var\left[\hat{\vartheta}_i^{(Ait)}\right]}$;
$G_{Var\left[\hat{\vartheta}_i^{(Ait)}\right]} = \exp(2G_{\mu_i} + G_{\sigma_i^2})\left[G_{\delta_i'}(1 - G_{\delta_i'}) + \frac{1}{2}G_{\delta_i'}(2G_{\sigma_i^2} + G_{\sigma_i^4})\right]/n_i$. Then, the 100(1- ζ)%FGCI for ϑ is

$$CI_{\vartheta}^{(fgci)} = [L_{\vartheta}^{(fgci)}, U_{\vartheta}^{(fgci)}] = [G_{\vartheta}(\zeta/2), G_{\vartheta}(1 - \zeta/2)] \tag{18}$$

where $G_{\vartheta}(\zeta)$ denotes the ζ^{th} percentiles of G_{ϑ}. Algorithm 1 shows the computational steps to obtain FGCI.

Algorithm 1: FGCI

1) Generate $V_i \sim N(0,1)$ and $U_i \sim \chi^2_{n_{i(1)}-1}$ are independent.
2) Compute the FGPQs G_{μ_i}, $G_{\sigma_i^2}$ and $G_{\delta_i'}$.
3) Compute G_{w_i} and G_{ϑ_i} leading to obtain G_{ϑ}.
4) Repeat steps 1-3, a number of times, m=2500, compute 95%FGCI for ϑ, as given in (18).

Large Sample Interval

Recall that the Aitchitson estimator is $\hat{\vartheta}_i^{(Ait)} = (1 - \hat{\delta}_i)\exp(\hat{\mu}_i)\psi_{n_{i(1)}}(\hat{\sigma}_i^2/2)$ and the variance of $\hat{\vartheta}_i^{(Ait)}$ is $Var\left[\hat{\vartheta}_i^{(Ait)}\right] = \exp(2\mu_i + \sigma_i^2)\left[\delta_i(1 - \delta_i) + \frac{1}{2}(1 - \delta_i)(2\sigma_i^2 + \sigma_i^4)\right]/n_i$. Replacing $\hat{\mu}_i$, $\hat{\sigma}_i^2$ and $\hat{\delta}_i$, the approximated variance are obtained. The pooled estimate of ϑ_i is

$$\hat{\vartheta} = \frac{\sum_{i=1}^{k} w_i \hat{\vartheta}_i^{(Ait)}}{\sum_{i=1}^{k} w_i} \tag{19}$$

where $w_i = 1/\widehat{Var}\left[\hat{\vartheta}_i^{(Ait)}\right]$. Hence, the $100(1 - \zeta)\%$ LS interval for ϑ is

$$CI_{\vartheta}^{(ls)} = [L_{\vartheta}^{(ls)}, U_{\vartheta}^{(ls)}] = \left[\hat{\vartheta} - z_{1-\frac{\zeta}{2}}\sqrt{1/\sum_{i=1}^{k} w_i}, \hat{\vartheta} + z_{1-\frac{\zeta}{2}}\sqrt{1/\sum_{i=1}^{k} w_i}\right] \tag{20}$$

where z_{ζ} denotes the ζ^{th} percentiles of standard normal $N(0,1)$. The LS interval can be estimated easily in Algorithm 2.

Algorithm 2: LS

1) Compute $\hat{\vartheta}_i^{(Ait)}$ and $\widehat{Var}\left[\hat{\vartheta}_i^{(Ait)}\right]$.
2) Compute $\hat{\vartheta}$.
3) Compute 95%LS interval for ϑ, as given in (20).

Method of Variance Estimates Recovery

This method produces a closed-form CI such that it is strong point and easy computation. For this reason, the MOVER interval for common delta-lognormal mean is considered under k individual random samples. The MOVER for linear combination of $\vartheta_i; i = 1, 2, ..., k$ is described. Let $\hat{\vartheta}_1, \hat{\vartheta}_2, ..., \hat{\vartheta}_k$ are independent unbiased estimators of $\vartheta_1, \vartheta_2, ..., \vartheta_k$, respectively. Also, let $[l_i, u_i]$ stands for the $100(1-\zeta)\%$CI for ϑ_i. According to Krishnamoorthy and Oral (2015), the $100(1-\zeta)\%$MOVER for $\sum_{i=1}^{k} c_i \vartheta_i$ is

$$\begin{aligned} CI_{\sum_{i=1}^{k} c_i\vartheta_i} &= [L_{\sum_{i=1}^{k} c_i\vartheta_i}, U_{\sum_{i=1}^{k} c_i\vartheta_i}] \\ &= \left[\sum_{i=1}^{k} c_i\hat{\vartheta}_i - \sqrt{\sum_{i=1}^{k} c_i^2\left(\hat{\vartheta}_i - l_i^*\right)^2}, \sum_{i=1}^{k} c_i\hat{\vartheta}_i + \sqrt{\sum_{i=1}^{k} c_i^2\left(\hat{\vartheta}_i - u_i^*\right)^2}\right] \end{aligned} \tag{21}$$

where $l_i^* = \begin{cases} l_i & ; c_i > 0 \\ u_i & ; c_i < 0 \end{cases}$ and $u_i^* = \begin{cases} u_i & ; c_i > 0 \\ l_i & ; c_i < 0 \end{cases}$. Next, the closed-form CIs for ϑ_i are needed for constructing MOVER for ϑ. The ϑ_i is log-transformed as

$$\ln \vartheta_i = \ln \delta_i^* + (\mu_i + \sigma_i^2) \tag{22}$$

where $\delta_i^* = 1 - \delta_i$. Let $\hat{\mu}_i$, $\hat{\sigma}_i^2$ and $\hat{\delta}^*$ be the unbiased estimates of μ_i, σ_i^2 and δ_i, respectively. Hasan and Krishnamoorthy (2018) presented the MOVER for single delta-lognormal mean so that the MOVER for ϑ_i is given by

$$\begin{aligned} L_{\vartheta_i} &= \exp\left\{\ln\hat{\delta}_i^* + (\hat{\mu}_i + \hat{\sigma}_i^2) - \sqrt{\left(\ln\hat{\delta}_i^* - l_{\ln\delta_i^*}\right)^2 + \left(\hat{\mu}_i + \hat{\sigma}_i^2 - l_{\mu_i+\sigma_i^2}\right)^2}\right\} \\ U_{\vartheta_i} &= \exp\left\{\ln\hat{\delta}_i^* + (\hat{\mu}_i + \hat{\sigma}_i^2) - \sqrt{\left(\ln\hat{\delta}_i^* - u_{\ln\delta_i^*}\right)^2 + \left(\hat{\mu}_i + \hat{\sigma}_i^2 - u_{\mu_i+\sigma_i^2}\right)^2}\right\} \end{aligned} \tag{23}$$

where

$$\begin{aligned} (l_{\ln\delta_i^*}, u_{\ln\delta_i^*}) &= \ln\left[\left(\hat{\delta}_i^* + \frac{T_{i,\zeta/2}^2}{2n_i} \mp T_{i,1-\zeta/2}\sqrt{\frac{\hat{\delta}_i^*(1-\hat{\delta}_i^*)}{n_i} + \frac{T_{i,\zeta/2}^2}{4n_i^2}}\right) / (1 + T_{i,\zeta/2}^2/n_i)\right] \\ (l_{\mu_i+\sigma_i^2}, u_{\mu_i+\sigma_i^2}) &= \left[(\hat{\mu}_i + \hat{\sigma}_i^2/2) - \left\{\left(\frac{Z_{i,\zeta/2}\hat{\sigma}_i^2}{n_{i(1)}}\right)^2 + \frac{\hat{\sigma}_i^4}{4}\left(1 - \frac{n_{i(1)}-1}{\chi^2_{i,1-\zeta/2,n_{i(1)}-1}}\right)^2\right\}^{1/2},\right. \\ &\qquad \left.(\hat{\mu}_i + \hat{\sigma}_i^2/2) + \left\{\left(\frac{Z_{i,\zeta/2}\hat{\sigma}_i^2}{n_{i(1)}}\right)^2 + \frac{\hat{\sigma}_i^4}{4}\left(\frac{n_{i(1)}-1}{\chi^2_{i,\zeta/2,n_{i(1)}-1}} - 1\right)^2\right\}^{1/2}\right] \end{aligned} \tag{24}$$

Note that both $T_i = (n_{i(1)} - n_i\delta^*)/\sqrt{n_i\delta_i^*(1-\delta_i^*)} \overset{d}{\sim} N(0,1)$, and $Z_i = (\hat{\mu}_i - \mu_i)/\sqrt{\hat{\sigma}_i^2/n_{i(1)}} \overset{d}{\sim} N(0,1)$ are independent random variables. According to Krishnamoorthy and Oral (2015), the $100(1-\zeta)\%$ MOVER interval for ϑ is

$$\begin{aligned} CI_{\vartheta}^{(mover)} &= [L_{\vartheta}, U_{\vartheta}] \\ &= \left[\frac{\sum_{i=1}^{k} w_i\hat{\vartheta}_i^{(Ait)}}{\sum_{i=1}^{k} w_i} - \sqrt{\frac{\sum_{i=1}^{k} w_i^2\left(\hat{\vartheta}_i^{(Ait)} - L_{\vartheta_i}\right)^2}{\sum_{i=1}^{k} w_i^2}}, \frac{\sum_{i=1}^{k} w_i\hat{\vartheta}_i^{(Ait)}}{\sum_{i=1}^{k} w_i} - \sqrt{\frac{\sum_{i=1}^{k} w_i^2\left(\hat{\vartheta}_i^{(Ait)} - U_{\vartheta_i}\right)^2}{\sum_{i=1}^{k} w_i^2}}\right] \end{aligned} \tag{25}$$

where $w_i = 1/\widehat{Var}\left[\hat{\vartheta}_i^{(Ait)}\right]$. Algorithm 3 describes the steps to construct the MOVER interval.

Algorithm 3: MOVER

1) Compute CIs for $\ln\delta_i^*$ and $\mu_i+\sigma_i^2$ are $(l_{\ln\delta_i^*}, u_{\ln\delta_i^*})$ and $(l_{\mu_i+\sigma_i^2}, u_{\mu_i+\sigma_i^2})$, respectively.
2) Compute MOVER for ϑ_i, as given in (23).
3) Compute 95%MOVER for ϑ, given in (25).

Parametric Bootstrap

This is expanded from parametric bootstrap on the common mean of several heterogeneous log-normal distributions, proposed by Malekzadeh and Kharrati-Kopaei (2019). The delta-lognormal mean is transformed by taking logarithm as

$$\mu_i = \ln\left(\frac{\vartheta}{1-\delta_i}\right) - \frac{\sigma_i^2}{2} \tag{26}$$

The likelihood of $(\vartheta, \sigma_i^2, \delta_i)$ is

$$L(\vartheta, \sigma_i^2, \delta_i|w_{ij}) = \prod_{i=1}^{k}\binom{n_i}{n_{i(0)}}\delta_i(1-\delta_i)\frac{1}{(2\pi\sigma_i^2)^{n_{i(1)}/2}}\exp\left\{-\frac{1}{2\sigma_i^2}\sum_{j=1}^{n_{i(1)}}\left(\ln w_{ij} - \ln\left(\frac{\vartheta}{1-\delta_i}\right)+\frac{\sigma_i^2}{2}\right)^2\right\} \tag{27}$$

which leads to obtain the MLEs of $\ln\vartheta$ and σ_i^2 are

$$\begin{aligned}\ln\hat{\vartheta}_{mle} &= \frac{\sum_{i=1}^{k}\hat{w}_{mle,i}\left[\hat{\mu}_i+\ln(1-\hat{\delta}_i)\right]+N/2}{\sum_{i=1}^{k}\hat{w}_{mle,i}}\\ \hat{\sigma}_{mle,i}^2 &= -2+2\sqrt{1+\hat{\sigma}_i^2+\{\hat{\mu}-\ln[\hat{\vartheta}/(1-\hat{\delta}_i)]\}^2}\end{aligned} \tag{28}$$

where $\hat{w}_{mle,i} = n_{i(1)}/\hat{\sigma}_{mle,i}^2$ and $\ln\hat{\vartheta} = \frac{\sum_{i=1}^{k}\hat{w}_i\left[\hat{\mu}_i+\ln(1-\hat{\delta}_i)\right]+N/2}{\sum_{i=1}^{k}\hat{w}_i}$; $\hat{w}_i = n_{i(1)}/\hat{\sigma}_i^2$. If $\delta_i = 0$, then it becomes the common lognormal mean; see detail in Krishnamoorthy and Oral (2015). Apply central limit theorem, we obtain that $\left(\ln\hat{\vartheta}_{mle} - \ln\vartheta\right)\sqrt{\sum_{i=1}^{k}\hat{w}_{mle,i}} \sim N(0,1)$ such that $T = \left(\ln\hat{\vartheta}_{mle}-\ln\vartheta\right)^2\sum_{i=1}^{k}\hat{w}_{mle,i} \sim \chi^2_{n_{i(1)}-1}$. It is well-known that $\hat{\mu}_i$, $\hat{\sigma}_i^2$ and $\hat{\delta}_i$ are independent random variables, and $\hat{\mu}_i \sim N(\ln\left(\frac{\vartheta}{1-\delta_i}\right)-\frac{\sigma_i^2}{2}, \sigma_i^2/n_{i(1)})$, $(n_{i(1)}-1)\hat{\sigma}_i^2/\sigma_i^2 \sim \chi^2_{n_{i(1)}-1}$ and $\hat{\delta}_i \sim N(\delta, \delta(1-\delta)/n_i)$ are obtained. Let $\eta = \mu_i + \sigma_i^2/2$ so that it can be written as $T = \frac{\sum_{i=1}^{k}\hat{w}_{mle,i}\left[\hat{\mu}_i+\ln(1-\hat{\delta}_i)-\eta-\ln(1-\delta_i)\right]+N/2}{\sum_{i=1}^{k}\hat{w}_{mle,i}}$. It can be seen that the distribution of T is complicated, although it possibly depends on the nuisance parameter σ_i^2's and δ_i's, but it does not depend on $\ln\vartheta$. Thus, the exact distribution of T is practically unknown. We propose the PB pivotal variable corresponding to T^{PB} as

$$T^{PB} = \left(\ln\hat{\vartheta}_{mle}^{PB} - \ln\hat{\vartheta}\right)^2\sum_{i=1}^{k}\hat{w}_{mle,i}^{PB} \tag{29}$$

where $\ln\hat{\vartheta}_{mle}^{PB} = \frac{\sum_{i=1}^{k}\hat{w}_{mle,i}^{PB}\left[\hat{\mu}_i^{PB}+\ln(1-\hat{\delta}_i^{PB})\right]+N/2}{\sum_{i=1}^{k}\hat{w}_{mle,i}^{PB}}$; $\hat{w}_i^{PB} = n_{i(1)}/\hat{\sigma}_i^{2B}$, $\hat{\mu}_i^{PB} \sim N(\hat{\mu}_i^B, \hat{\sigma}_i^{2B}/n_{i(1)})$, $\hat{\sigma}_i^{2PB} \sim \hat{\sigma}_i^{B2}\chi^2_{n_{i(1)}-1}$ $/(n_{i(1)}-1)$ and $\hat{\delta}^{PB} \sim beta(n_{i(0)}^B+0.5, n_{i(1)}^B+0.5)$; $n_{i(0)}^B = n_i\hat{\delta}_i^B$ and $n_{i(1)}^B = n_i - n_{i(0)}^B$. Note that $\hat{\mu}_i^B$, $\hat{\sigma}_i^{2B}$, $\hat{\delta}_i^B$ be observed values of $\hat{\mu}_i$, $\hat{\sigma}_i^2$, $\hat{\delta}_i$ from random sampling with replacement based on bootstrap approach. Then, the $100(1-\zeta)$%PB interval for ϑ is

$$CI_{\vartheta}^{(pb)} = \exp\left[\ln\hat{\vartheta}_{mle} \mp \sqrt{q_{\zeta}^{PB}/\sum_{i=1}^{k}\hat{w}_{mle,i}}\right] \tag{30}$$

where q_{ζ}^{PB} denotes the $(1-\zeta)^{th}$ percentile of distribution of T^{PB}. The PB interval can be constructed in Algorithm 4.

Algorithm 4: PB

1) Compute $\hat{\mu}_i$, $\hat{\sigma}_i^2$ and $\hat{\delta}$ leading to obtain $\ln\hat{\vartheta}$.
2) Compute $\ln\hat{\vartheta}_{mle}$ and $\hat{\sigma}^2_{mle,i}$.
3) Generate $\hat{\mu}_i^{PB}$, $\hat{\sigma}_i^{2PB}$ and $\hat{\delta}_i^{PB}$ leading to compute $\ln\hat{\vartheta}^{PB}_{mle}$.
4) Repeat steps 1-3, a number of time $m = 2500$, compute T^{PB} to obtain q_ζ^{PB}.
5) Compute 95%PB interval for ϑ, as given in (30).

Highest Posterior Density Intervals

The HPD interval is constructed from the posterior distribution, defined by Box and Tiao (1973). Note that the prior of ϑ_i is updated with its likelihood function leading to obtain the posterior distribution based on Bayesian approach. Recall that $W_{ij} \sim \Delta(\mu_i, \sigma_i^2, \delta_i)$. The likelihood is

$$P(w_{ij}|\mu_i,\sigma_i^2,\delta_i) \propto \prod_{i=1}^{k} \delta_i^{n_{i(0)}} (1-\delta_i)^{n_{i(1)}} (\sigma_i^2)^{-n_{i(1)}/2} \exp\left\{-\frac{1}{2\sigma_i^2}\sum_{j=1}^{n_{i(1)}}\left(\ln w_{ij}-\mu_i\right)^2\right\} \tag{31}$$

For k individual samples, Miroshnikov et al. (2015) described the pooled independent sub-posterior samples toward the joint posterior distributions ϑ are combined using weighted averages as

$$\vartheta^{post} = \sum_{i=1}^{k} w_i \vartheta_i^{post}\left(\sum_{i=1}^{k} w_i\right)^{-1} \tag{32}$$

where ϑ_i^{post} be the posterior samples of ϑ_i; $i = 1,2,\ldots,k$. The inversely sample variance is weighted the posterior based on i^{th} samples, denoted $w_i = Var^{-1}(\hat{\vartheta}_i|w_{ij})$. There are the HPD-based on different priors developed for estimating the common delta-lognormal mean as follows.

Jeffreys' Rule Prior

Harvey and van der Merwe (2012) defined this prior as

$$P(\vartheta)_{JR} \propto \prod_{i=1}^{k} \sigma_i^{-3}\delta_i^{-1/2}(1-\delta_i)^{1/2} \tag{33}$$

which is combined with the likelihood (34) to obtain the posterior of ϑ as

$$\begin{aligned} P(w_{ij}|\vartheta) &\propto \prod_{i=1}^{k} \delta_i^{n_{i(0)}-1/2}(1-\delta_i)^{n_{i(1)}+1/2}(\sigma_i^2)^{-(n_{i(1)}+3)/2}\exp\left\{-\frac{1}{2\sigma_i^2}\sum_{j=1}^{n_{i(1)}}\left(\ln w_{ij}-\mu_i\right)^2\right\} \\ &\propto \prod_{i=1}^{k} \delta_i^{(n_{i(0)}+1/2)-1}(1-\delta_i)^{(n_{i(1)}+3/2)-1}(\sigma_i^2)^{-\frac{(n_{i(1)}+1)}{2}-1} \\ &\quad \exp\left\{-\frac{1}{2\sigma_i^2}\left[(n_{i(1)}-1)\hat{\sigma}_i^2 + n_{i(1)}(\hat{\mu}_i-\mu_i)^2\right]\right\} \end{aligned} \tag{34}$$

This leads to obtain the marginal posterior distributions of μ_i, σ_i^2 and δ_i are

$$\begin{aligned} \mu_i^{(JR)}|\sigma^2_{i,JR}, w_{ij} &\sim N\left(\hat{\mu}_i, \sigma_i^{2(JR)}/n_{i(1)}\right) \\ \sigma_i^{2(JR)}|w_{ij} &\sim IG\left((n_{i(1)}+1)/2, (n_{i(1)}+1)\hat{\sigma}_i^2/2\right) \\ \delta_i^{(JR)}|w_{ij} &\sim beta\left(n_{i(0)}+1/2, n_{i(1)}+3/2\right) \end{aligned} \tag{35}$$

The pooled posterior of ϑ is weighted by its inversely estimated variance as

$$\vartheta^{post} = \sum_{i=1}^{k} w_i^{(JR)} \vartheta_i^{(JR)_p} \left(\sum_{i=1}^{k} w_i^{(JR)} \right)^{-1} \tag{36}$$

where

$\vartheta_i^{(JR)_p} = (1-\delta_i^{(JR)}) \exp(\mu_i^{(JR)} + \sigma_i^{2(JR)}/2)$

$w_i^{(JR)} = \left\{ n_i^{-1} \exp(2\mu_i^{(JR)} + \sigma_i^{2(JR)}) \left[\delta_i^{(JR)}(1-\delta_i^{(JR)}) + \frac{1}{2}(1-\delta_i^{(JR)})(2\sigma_i^{2(JR)} + \sigma_i^{4(JR)}) \right] \right\}^{-1}$

From (36), the $100(1-\zeta)\%$HPD-based Jeffreys' rule prior for ϑ is constructed.

Normal-Gamma-Beta Prior

Maneerat et al. (2020) proposed the HPD-based on normal-gamma prior for the ratio of delta-lognormal variances, while its performance worked well than HPD-based Jeffreys' rule prior of Harvey and van der Merwe (2012). Suppose that $\boldsymbol{Y} = \ln \boldsymbol{W}$ be a random variable of normal distribution with mean $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_k)$ and precision $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_k)$ where $\boldsymbol{W} \sim LN(\boldsymbol{\mu}, \boldsymbol{\lambda})$ and $\lambda_i = \sigma_i^{-2}$. The normal-gamma-beta prior of $\vartheta = (\mu_i, \lambda_i, \delta_i)'$ is defined as

$$P(\vartheta) \propto \prod_{i=1}^{k} \lambda_i^{-1} [\delta_i(1-\delta_i)]^{-1/2} \tag{37}$$

where (μ_i, λ_i) has a normal-gamma distribution, while δ_i has a beta distribution, denoted as $(\mu_i, \lambda_i) \sim NG(\mu_i, \lambda_i | \mu, k_{i(0)} = 0, \alpha_{i(0)} = -1/2, \beta_{i(0)} = 0)$ and $\delta_i \sim beta(1/2, 1/2)$, respectively. When the prior (37) is combined with the likelihood (34), then the posterior density of ϑ is

$$\begin{aligned} P(\vartheta | w_{ij}) \quad \propto \quad & \prod_{i=1}^{k} \delta_i^{n_{i(0)}-1/2} (1-\delta_i)^{n_{i(1)}-1/2} \lambda_i^{\frac{n_{i(1)}-1}{2}-1} \exp\left\{ -\frac{\lambda_i}{2} \sum_{j=1}^{n_{i(1)}} (\ln w_{ij} - \hat{\mu}_i)^2 \right\} \lambda_i^{1/2} \\ & \exp\left\{ -\frac{n_{i(1)}\lambda_i}{2} (\mu_i - \mu_i^*)^2 \right\} \end{aligned} \tag{38}$$

which is integrated out to obtain the marginal posterior distributions of μ_i, λ_i and δ_i are

$$\begin{aligned} & \mu_i^{(NGB)} | w_{ij} \sim t_{df} \left(\mu_i | \hat{\mu}_i, \sum_{j=1}^{n_{i(1)}} (\ln w_{ij} - \hat{\mu}_i)^2 / [n_{i(1)}(n_{i(1)} - 1)] \right) \\ & \lambda_i^{(NGB)} | w_{ij} \sim G \left(\lambda_i | (n_{i(1)} - 1)/2, \sum_{j=1}^{n_{i(1)}} (\ln w_{ij} - \hat{\mu}_i)^2 / 2 \right) \\ & \delta_i^{(NGB)} | w_{ij} \sim beta\left(n_{i(0)} + 1/2, n_{i(1)} + 1/2 \right) \end{aligned} \tag{39}$$

where $df = 2(n_{i(1)} - 1)$ and $\sigma_i^{2(NGB)} | w_{ij} \sim IG\left(\sigma_i^2 | (n_{i(1)} - 1)/2, \sum_{j=1}^{n_{i(1)}} (\ln w_{ij} - \hat{\mu}_i)^2 / 2 \right)$. Similarly, the pooled posterior of ϑ based on is

$$\vartheta^{post} = \sum_{i=1}^{k} w_i^{(NGB)} \vartheta_i^{(NGB)_p} \left(\sum_{i=1}^{k} w_i^{(NGB)} \right)^{-1} \tag{40}$$

where

$\vartheta_i^{(NGB)_p} = (1-\delta_i^{(NGB)}) \exp(\mu_i^{(NGB)} + \sigma_i^{2(NGB)}/2)$

$w_i^{(NGB)} = \left\{ n_i^{-1} \exp(2\mu_i^{(NGB)} + \sigma_i^{2(NGB)}) \left[\delta_i^{(NGB)}(1-\delta_i^{(NGB)}) \frac{1}{2}(1-\delta_i^{(NGB)})(2\sigma_i^{2(NGB)} + \sigma_i^{4(NGB)}) \right] \right\}^{-1}$

From (40), the $100(1-\zeta)\%$HPD-based normal-gamma-beta prior for ϑ is constructed. Algorithm 5 details the steps to construct the HPD-JR and HPD-NGB.

Table 1. Parameter settings for sample cases $k = 2, 5, 10$

Scenarios	$(n_1,...,n_k)$	$(\delta_1,...,\delta_k)$	$(\sigma_1^2,...,\sigma_k^2)$
$k = 2$			
1-9	(30,30)	(0.1,0.2), (0.2,0.5), (0.3,0.7)	(1,2), (2,4), (3,5)
10-18	(30,50)	(0.1,0.2), (0.2,0.5), (0.3,0.7)	(1,2), (2,4), (3,5)
19-27	(50,50)	(0.1,0.2), (0.2,0.5), (0.3,0.7)	(1,2), (2,4), (3,5)
28-36	(50,100)	(0.1,0.2), (0.2,0.5), (0.3,0.7)	(1,2), (2,4), (3,5)
37-45	(100,100)	(0.1,0.2), (0.2,0.5), (0.3,0.7)	(1,2), (2,4), (3,5)
$k = 5$			
45-54	(30_5)	$(0.05, 0.1_2, 0.2_2)$, $(0.2_2, 0.4_3)$, $(0.5_2, 0.7_3)$	$(1_2, 2_3)$, $(2_2, 3_3)$, $(3_2, 5_3)$
55-63	$(30_2, 50_3)$	$(0.05, 0.1_2, 0.2_2)$, $(0.2_2, 0.4_3)$, $(0.5_2, 0.7_3)$	$(1_2, 2_3)$, $(2_2, 3_3)$, $(3_2, 5_3)$
64-72	$(30_2, 50_2, 100_1)$	$(0.05, 0.1_2, 0.2_2)$, $(0.2_2, 0.4_3)$, $(0.5_2, 0.7_3)$	$(1_2, 2_3)$, $(2_2, 3_3)$, $(3_2, 5_3)$
73-81	$(30, 50_2, 100_2)$	$(0.05, 0.1_2, 0.2_2)$, $(0.2_2, 0.4_3)$, $(0.5_2, 0.7_3)$	$(1_2, 2_3)$, $(2_2, 3_3)$, $(3_2, 5_3)$
82-90	(50_5)	$(0.05, 0.1_2, 0.2_2)$, $(0.2_2, 0.4_3)$, $(0.5_2, 0.7_3)$	$(1_2, 2_3)$, $(2_2, 3_3)$, $(3_2, 5_3)$
91-99	$(50_2, 100_3)$	$(0.05, 0.1_2, 0.2_2)$, $(0.2_2, 0.4_3)$, $(0.5_2, 0.7_3)$	$(1_2, 2_3)$, $(2_2, 3_3)$, $(3_2, 5_3)$
100-108	(100_5)	$(0.05, 0.1_2, 0.2_2)$, $(0.2_2, 0.4_3)$, $(0.5_2, 0.7_3)$	$(1_2, 2_3)$, $(2_2, 3_3)$, $(3_2, 5_3)$
$k = 10$			
109-114	$(30_5, 50_5)$	$(0.1_5, 0.2_5)$, $(0.2_5, 0.5_5)$	$(1_5, 2_5)$, $(2_5, 4_5)$, $(3_5, 5_5)$
115-120	$(30_3, 50_3, 100_4)$	$(0.1_5, 0.2_5)$, $(0.2_5, 0.5_5)$	$(1_5, 2_5)$, $(2_5, 4_5)$, $(3_5, 5_5)$
121-126	$(50_5, 100_5)$	$(0.1_5, 0.2_5)$, $(0.2_5, 0.5_5)$	$(1_5, 2_5)$, $(2_5, 4_5)$, $(3_5, 5_5)$

Note: (30_5) stands for $(30, 30, 30, 30, 30)$.

Algorithm 5: HPD-JR and HPD-NGB

1) Compute $\hat{\mu}_i$, $\hat{\sigma}_i^2$ and $\hat{\delta}$.
2) Generate the posterior densities of μ_i, σ_i^2 and δ_i based-Jeffreys' rule (JR) and normal-gamma-beta (NGB) priors, as given in (35) and (39), respectively.
3) Compute the pooled posterior of ϑ based on JR and NGB priors, as given in (36) and (40), respectively.
4) Compute 95%HPD-JR and HPD-NGB for ϑ, defined by Box and Tiao (1973).

SIMULATION STUDIES AND RESULTS

The performances of the CIs were assessed by comparing their coverage probabilities (CPs) and average length (ALs) using Monte Carlo simulation. The best-performing CI is the one where the CP is closed to or greater than the nominal confidence level $1-\zeta$ and the AL also has the narrowest width. The CIs for the common delta-lognormal mean constructed using FGCI, LS, MOVER, PB, HPD-JR, and HPD-NGB were assessed in the study. The parameter settings are provided in Table 1. The number of generated random samples was fixed at 5000. For FGCI, the number of FGPQ was 2500 for each set of 5000 random samples. Algorithm 6 shows the computational steps to estimate the CP and AL performances of all of the methods.

Algorithm 6:

1) Generate $w_{ij} \sim \Delta(\mu_i, \sigma_i^2, \delta_i)$.
2) Compute the unbiased estimates $\hat{\mu}_i$, $\hat{\sigma}_i^2$ and $\hat{\delta}$.
3) Compute 95%CIs for ϑ based on FGCI, LS, MOVER, PB and HPDs in Algorithms 1, 2, 3, 4 and 5, respectively.
4) Repeat steps 1-3, a number of times $M = 5000$, the CP and AL are obtained each method.

The numerical results are summarized in terms of CP and AL for the CI performances in various sample cases. For $k = 2$ (Table 2 and Figure 1), FGCI performed well for all small-to-moderate sample sizes, as well as for large σ_i^2 and a moderate-to-large sample size. HPD-NGB attained stable and the best CP and AL for small σ_i^2 and a moderate-to-large sample size. MOVER and PB attained correct CPs but wider ALs than the other methods whereas LS and HPD-JR had lower CPs and smaller ALs. For $k = 5$ (Tables 3-4 and Figure 2),there were only two methods producing better CPs than the other methods in the various situations: MOVER (small δ_i and σ_i^2) and PB (large δ_i and σ_i^2). Moreover, the results were similar for $k = 10$ (Table 4 and Figure 3).

Table 2. Performance measures of 95%CIs for ϑ: 2,5,10 sample cases

Scenarios	CP						AL					
	FG	LS	MO	PB	HJ	HN	FG	LS	MO	PB	HJ	HN
$k = 2$												
1	0.959	0.897	0.967	0.994	0.916	0.941	1.556	1.296	2.005	2.324	1.353	1.436
2	0.958	0.857	0.947	0.996	0.924	0.941	5.169	3.770	7.287	8.631	4.186	4.335
3	0.963	0.821	0.959	0.996	0.919	0.932	13.088	8.675	23.312	22.883	9.905	10.220
4	0.962	0.886	0.978	0.995	0.917	0.939	1.487	1.211	2.181	2.155	1.247	1.386
5	0.953	0.832	0.962	0.995	0.913	0.922	4.875	3.487	9.881	7.818	3.811	4.066
6	0.951	0.793	0.971	0.991	0.901	0.912	12.311	7.740	37.615	21.129	8.875	9.378
7	0.961	0.829	0.972	0.982	0.920	0.940	1.511	1.095	3.968	2.173	1.224	1.406
8	0.950	0.778	0.974	0.995	0.900	0.911	4.821	3.123	293.620	7.649	3.566	3.916
9	0.939	0.725	0.973	0.988	0.866	0.887	13.159	7.067	8.0e4	23.632	8.680	9.419
10	0.960	0.900	0.965	0.992	0.915	0.941	1.503	1.249	1.936	2.225	1.362	1.395
11	0.961	0.848	0.941	0.992	0.924	0.940	5.128	3.712	6.765	8.667	4.298	4.368
12	0.965	0.819	0.952	0.998	0.919	0.931	12.297	8.382	20.057	21.597	9.819	9.894
13	0.960	0.896	0.977	0.992	0.917	0.942	1.366	1.147	1.909	2.004	1.203	1.271
14	0.961	0.851	0.964	0.996	0.916	0.931	4.593	3.422	7.236	7.458	3.761	3.889
15	0.949	0.790	0.958	0.994	0.894	0.905	11.116	7.517	22.293	19.310	8.507	8.718
16	0.963	0.860	0.972	0.974	0.928	0.943	1.354	1.033	2.141	1.928	1.155	1.257
17	0.952	0.803	0.976	0.992	0.900	0.917	4.397	3.048	10.772	6.889	3.418	3.630
18	0.940	0.737	0.968	0.989	0.872	0.889	11.065	6.663	43.755	19.011	7.903	8.247
19	0.961	0.914	0.966	0.992	0.921	0.946	1.153	1.009	1.382	1.696	1.043	1.076
20	0.965	0.895	0.946	0.991	0.938	0.949	3.668	2.924	4.309	5.981	3.178	3.229
21	0.962	0.863	0.952	0.996	0.930	0.940	8.747	6.665	11.805	14.651	7.272	7.395
22	0.958	0.910	0.978	0.985	0.919	0.944	1.091	0.945	1.414	1.555	0.945	1.031
23	0.965	0.883	0.969	0.996	0.926	0.937	3.336	2.695	4.578	5.204	2.811	2.950
24	0.961	0.840	0.972	0.995	0.921	0.928	7.887	5.987	13.164	12.757	6.338	6.605
25	0.969	0.868	0.980	0.958	0.930	0.953	1.120	0.866	1.610	1.503	0.937	1.070
26	0.954	0.839	0.970	0.997	0.916	0.926	3.208	2.433	6.544	4.735	2.621	2.830
27	0.946	0.773	0.970	0.992	0.893	0.903	7.803	5.443	26.105	12.382	6.011	6.376
28	0.958	0.912	0.972	0.979	0.916	0.947	1.119	0.952	1.397	1.615	1.054	1.051
29	0.956	0.872	0.921	0.958	0.927	0.943	3.745	2.836	4.238	6.098	3.338	3.330
30	0.961	0.846	0.937	0.987	0.925	0.936	8.488	6.274	10.833	13.991	7.332	7.320
31	0.962	0.927	0.985	0.978	0.919	0.949	0.984	0.876	1.322	1.433	0.908	0.929
32	0.960	0.880	0.958	0.992	0.925	0.940	3.214	2.618	4.169	5.150	2.818	2.860
33	0.958	0.838	0.960	0.994	0.910	0.925	7.360	5.744	10.824	12.105	6.256	6.279
34	0.963	0.888	0.977	0.922	0.938	0.954	0.975	0.785	1.322	1.352	0.876	0.922
35	0.958	0.860	0.971	0.995	0.917	0.929	2.915	2.343	4.321	4.424	2.486	2.586
36	0.951	0.820	0.973	0.995	0.901	0.916	6.726	5.103	11.951	10.823	5.511	5.626
37	0.957	0.935	0.960	0.970	0.927	0.948	0.802	0.722	0.923	1.168	0.743	0.753
38	0.955	0.916	0.926	0.953	0.942	0.948	2.442	2.044	2.541	3.935	2.220	2.219
39	0.957	0.888	0.939	0.981	0.937	0.945	5.608	4.594	6.295	9.049	4.984	4.998
40	0.961	0.942	0.975	0.957	0.924	0.954	0.740	0.679	0.911	1.062	0.659	0.702
41	0.961	0.920	0.960	0.988	0.933	0.950	2.199	1.925	2.558	3.401	1.958	2.012
42	0.955	0.875	0.960	0.994	0.925	0.931	4.976	4.209	6.298	7.813	4.318	4.439
43	0.967	0.909	0.980	0.863	0.937	0.960	0.773	0.625	0.972	1.012	0.659	0.743
44	0.960	0.896	0.970	0.993	0.928	0.939	2.076	1.750	2.684	3.013	1.788	1.921
45	0.952	0.835	0.970	0.996	0.908	0.914	4.683	3.786	7.007	7.008	3.952	4.182

Notes: FG, fiducial generalized confidence interval; MO, method of variance estimates recovery; HJ, HPD-based Jeffreys' rule prior, HPD-JR; HN, HPD-based normal-gamma-beta prior.

Table 3. Continued.

Scenarios	CP						AL					
	FG	LS	MO	PB	HJ	HN	FG	LS	MO	PB	HJ	HN
$k = 5$												
46	0.885	0.790	0.988	0.989	0.757	0.846	0.963	0.819	1.794	1.532	0.848	0.956
47	0.789	0.627	0.973	0.996	0.674	0.715	2.240	1.908	4.982	3.897	1.991	2.176
48	0.840	0.613	0.953	0.997	0.723	0.746	5.325	4.529	13.769	12.250	4.744	4.870
49	0.894	0.800	0.993	0.978	0.779	0.864	0.900	0.765	1.825	1.439	0.773	0.905
50	0.783	0.623	0.972	0.998	0.680	0.711	2.008	1.711	5.203	3.608	1.750	1.955
51	0.797	0.580	0.959	0.996	0.680	0.701	4.700	4.066	16.626	11.353	4.118	4.287
52	0.893	0.735	0.989	0.896	0.816	0.853	0.753	0.589	2.849	1.433	0.636	0.764
53	0.768	0.517	0.977	0.997	0.666	0.676	1.474	1.168	19.967	3.364	1.282	1.406
54	0.742	0.467	0.983	0.996	0.624	0.629	3.250	2.654	1.5e4	11.238	2.817	2.855
55	0.884	0.779	0.988	0.979	0.743	0.846	0.940	0.777	1.739	1.434	0.857	0.930
56	0.806	0.645	0.973	0.995	0.681	0.740	2.204	1.822	4.586	3.561	2.045	2.141
57	0.858	0.622	0.949	0.986	0.725	0.771	5.620	4.542	12.575	12.073	5.122	5.162
58	0.901	0.827	0.995	0.962	0.770	0.870	0.845	0.728	1.699	1.326	0.771	0.841
59	0.793	0.644	0.978	0.997	0.675	0.726	1.904	1.629	4.351	3.262	1.750	1.850
60	0.825	0.605	0.952	0.997	0.710	0.734	4.753	4.058	12.745	10.793	4.373	4.353
61	0.905	0.785	0.992	0.822	0.809	0.865	0.685	0.564	1.632	1.219	0.620	0.686
62	0.786	0.578	0.969	0.993	0.683	0.704	1.368	1.142	4.477	2.775	1.260	1.309
63	0.755	0.496	0.963	0.998	0.639	0.637	3.177	2.714	18.995	8.911	2.884	2.822
64	0.892	0.787	0.991	0.970	0.737	0.858	0.928	0.751	1.740	1.364	0.872	0.919
65	0.822	0.647	0.975	0.996	0.673	0.763	2.168	1.738	4.371	3.326	2.047	2.114
66	0.852	0.593	0.943	0.981	0.715	0.767	5.710	4.413	12.195	11.422	5.267	5.278
67	0.905	0.827	0.996	0.949	0.768	0.873	0.816	0.697	1.637	1.256	0.770	0.811
68	0.801	0.654	0.979	0.995	0.683	0.737	1.839	1.549	4.069	3.016	1.753	1.797
69	0.821	0.595	0.947	0.994	0.693	0.733	4.806	3.976	12.174	10.326	4.431	4.432
70	0.917	0.803	0.994	0.775	0.817	0.886	0.650	0.539	1.499	1.133	0.616	0.650
71	0.804	0.612	0.973	0.992	0.692	0.730	1.310	1.094	3.962	2.543	1.236	1.262
72	0.756	0.502	0.958	0.997	0.631	0.646	3.158	2.695	16.604	8.356	2.888	2.835
73	0.924	0.832	0.994	0.942	0.772	0.893	0.822	0.673	1.505	1.186	0.856	0.808
74	0.853	0.699	0.985	0.990	0.696	0.798	1.971	1.589	3.823	2.899	2.000	1.923
75	0.883	0.652	0.952	0.945	0.755	0.817	5.330	4.072	9.997	9.911	5.224	4.974
76	0.924	0.857	0.997	0.913	0.771	0.901	0.723	0.626	1.418	1.088	0.746	0.715
77	0.826	0.695	0.986	0.989	0.689	0.767	1.670	1.406	3.476	2.610	1.692	1.632
78	0.854	0.638	0.955	0.984	0.718	0.771	4.456	3.628	9.715	8.788	4.311	4.160
79	0.930	0.846	0.998	0.683	0.811	0.900	0.581	0.486	1.253	0.964	0.586	0.580
80	0.830	0.658	0.981	0.980	0.705	0.762	1.215	1.019	3.179	2.168	1.225	1.181
81	0.788	0.555	0.967	0.997	0.675	0.689	2.992	2.554	11.873	7.026	2.927	2.738
82	0.915	0.844	0.993	0.964	0.788	0.889	0.769	0.662	1.337	1.158	0.692	0.753
83	0.858	0.735	0.982	0.993	0.741	0.804	1.882	1.599	3.605	2.920	1.698	1.825
84	0.886	0.705	0.969	0.981	0.782	0.827	4.650	3.895	8.767	9.068	4.208	4.335
85	0.925	0.865	0.998	0.939	0.803	0.897	0.707	0.618	1.315	1.068	0.618	0.700
86	0.834	0.705	0.987	0.994	0.735	0.775	1.683	1.439	3.493	2.683	1.482	1.642
87	0.855	0.684	0.968	0.994	0.751	0.783	4.027	3.489	8.924	8.068	3.613	3.766
88	0.929	0.824	0.994	0.677	0.835	0.903	0.611	0.495	1.322	0.993	0.515	0.616
89	0.823	0.627	0.981	0.985	0.729	0.749	1.284	1.045	3.692	2.296	1.121	1.250
90	0.799	0.578	0.972	0.997	0.699	0.705	2.875	2.453	13.603	6.644	2.519	2.641
91	0.927	0.831	0.997	0.906	0.777	0.898	0.753	0.614	1.389	1.064	0.703	0.735
92	0.871	0.731	0.988	0.986	0.720	0.820	1.821	1.466	3.459	2.601	1.721	1.769
93	0.905	0.693	0.957	0.897	0.791	0.852	5.015	3.768	8.461	8.829	4.621	4.690
94	0.931	0.879	0.999	0.873	0.781	0.909	0.651	0.571	1.279	0.972	0.608	0.639
95	0.847	0.738	0.991	0.986	0.719	0.797	1.541	1.313	3.117	2.351	1.447	1.499
96	0.875	0.679	0.966	0.969	0.760	0.806	4.125	3.374	8.002	7.707	3.808	3.865
97	0.935	0.866	0.998	0.541	0.832	0.911	0.529	0.450	1.097	0.856	0.493	0.523
98	0.848	0.697	0.986	0.971	0.735	0.782	1.126	0.956	2.572	1.916	1.060	1.091
99	0.817	0.613	0.963	0.994	0.698	0.725	2.784	2.418	7.510	6.042	2.565	2.571

Notes: FG, fiducial generalized confidence interval; MO, method of variance estimates recovery; HJ, HPD-based Jeffreys' rule prior, HPD-JR; HN, HPD-based normal-gamma-beta prior.

Table 4. Continued.

Scenarios	CP						AL					
	FG	LS	MO	PB	HJ	HN	FG	LS	MO	PB	HJ	HN
$k = 5$												
100	0.941	0.888	0.998	0.863	0.813	0.920	0.557	0.484	0.954	0.806	0.510	0.536
101	0.906	0.827	0.995	0.973	0.799	0.875	1.413	1.201	2.515	2.029	1.288	1.361
102	0.929	0.790	0.975	0.861	0.845	0.889	3.639	2.946	5.529	6.174	3.365	3.428
103	0.948	0.923	1.000	0.801	0.816	0.931	0.501	0.456	0.909	0.741	0.452	0.487
104	0.888	0.816	0.996	0.978	0.784	0.853	1.253	1.095	2.373	1.852	1.121	1.216
105	0.905	0.775	0.981	0.953	0.822	0.859	3.147	2.678	5.326	5.441	2.893	2.975
106	0.955	0.907	0.999	0.289	0.852	0.943	0.438	0.372	0.838	0.668	0.373	0.433
107	0.881	0.761	0.994	0.939	0.781	0.833	0.992	0.823	2.044	1.536	0.863	0.972
108	0.868	0.722	0.984	0.987	0.781	0.805	2.331	2.005	5.072	4.308	2.088	2.208
$k = 10$												
109	0.728	0.675	0.998	0.927	0.566	0.692	0.612	0.501	1.554	0.932	0.545	0.623
110	0.661	0.500	0.979	0.891	0.570	0.588	1.644	1.291	3.867	3.278	1.500	1.637
111	0.504	0.352	0.950	0.978	0.481	0.404	3.159	2.561	8.645	7.286	2.996	3.076
112	0.720	0.692	0.999	0.904	0.587	0.690	0.557	0.459	1.519	0.832	0.483	0.574
113	0.532	0.452	0.976	0.985	0.512	0.462	1.393	1.159	3.853	2.682	1.260	1.404
114	0.361	0.290	0.955	0.998	0.403	0.274	2.556	2.218	8.570	5.943	2.411	2.505
115	0.789	0.723	0.999	0.808	0.561	0.762	0.554	0.440	1.416	0.789	0.546	0.560
116	0.716	0.524	0.985	0.578	0.590	0.653	1.635	1.180	3.478	2.915	1.559	1.624
117	0.593	0.406	0.964	0.872	0.519	0.507	3.289	2.406	7.754	6.380	3.189	3.214
118	0.782	0.773	1.000	0.780	0.586	0.758	0.477	0.404	1.317	0.696	0.474	0.483
119	0.626	0.514	0.988	0.947	0.535	0.561	1.337	1.076	3.348	2.360	1.284	1.341
120	0.447	0.347	0.965	0.992	0.450	0.355	2.570	2.108	7.290	5.180	2.506	2.531
121	0.826	0.773	1.000	0.736	0.592	0.796	0.488	0.399	1.266	0.695	0.444	0.486
122	0.774	0.620	0.994	0.438	0.647	0.720	1.460	1.086	3.072	2.512	1.328	1.438
123	0.659	0.460	0.977	0.798	0.553	0.577	3.002	2.236	6.597	5.502	2.775	2.921
124	0.828	0.826	1.000	0.708	0.606	0.802	0.426	0.368	1.187	0.615	0.387	0.427
125	0.688	0.595	0.995	0.912	0.591	0.627	1.205	0.992	2.912	2.039	1.094	1.197
126	0.520	0.426	0.979	0.984	0.486	0.439	2.390	1.989	6.222	4.479	2.224	2.344

Notes: FG, fiducial generalized confidence interval; MO, method of variance estimates recovery; HJ, HPD-based Jeffreys' rule prior, HPD-JR; HN, HPD-based normal-gamma-beta prior.

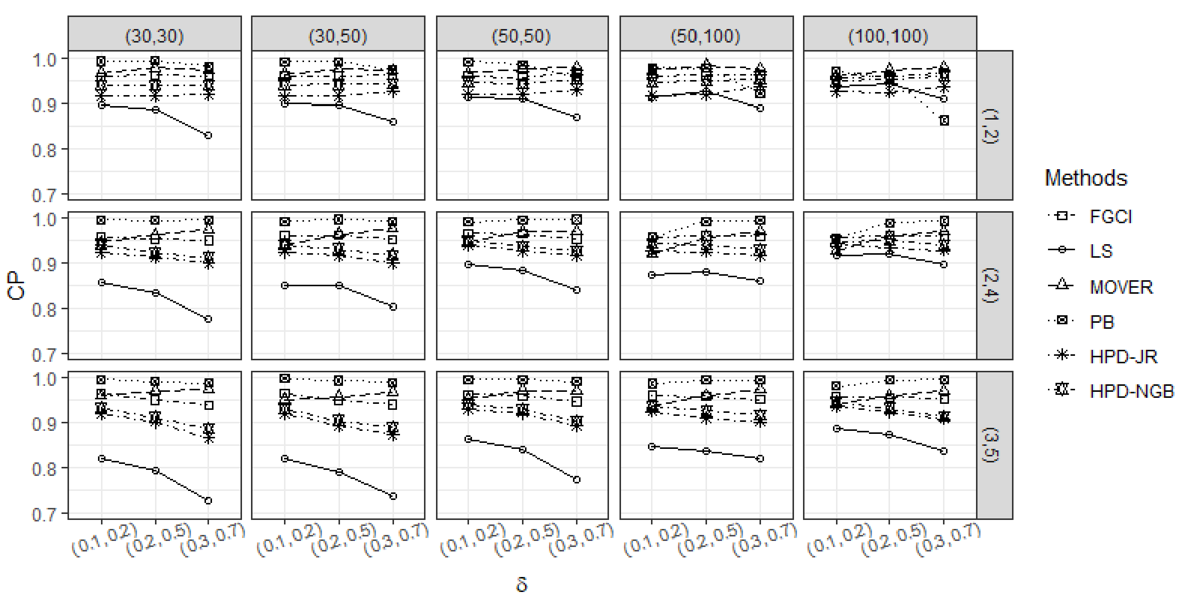

Figure 1. CP performances of 95%CIs for ϑ: 2 sample cases

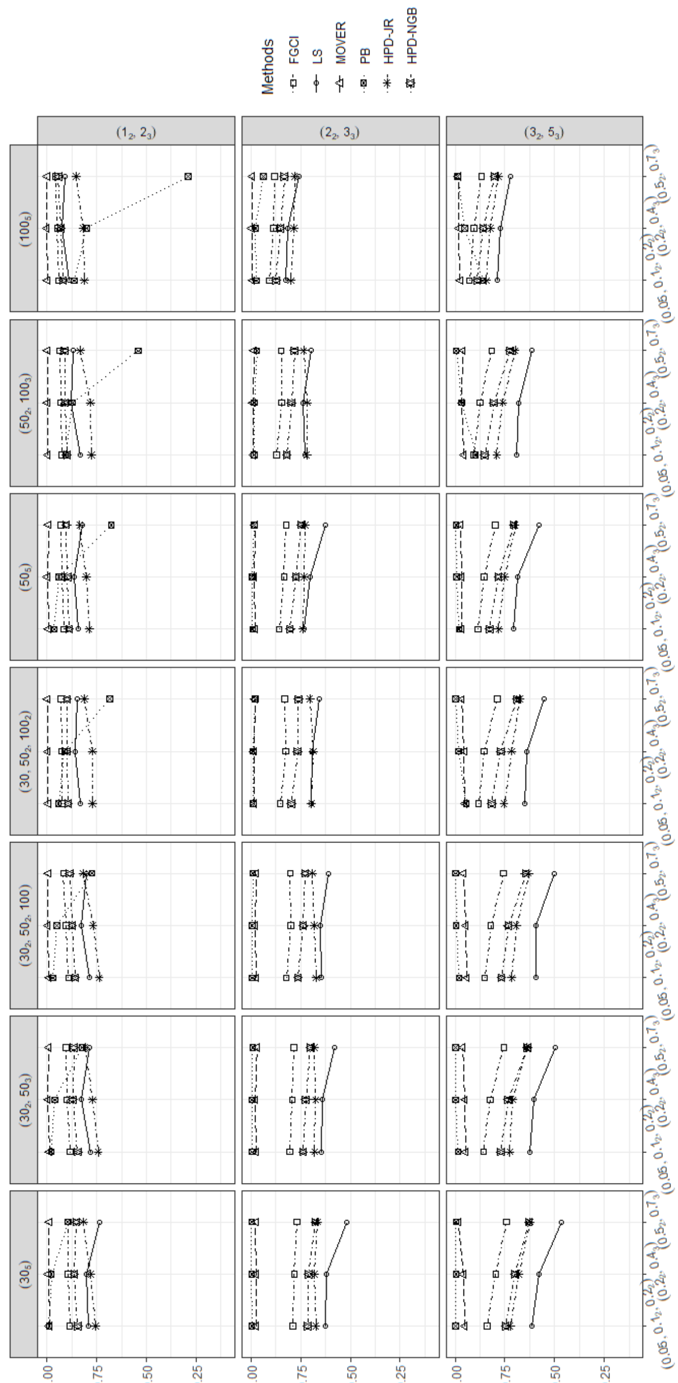

Figure 2. CP performances of 95%CIs for ϑ: 5 sample cases

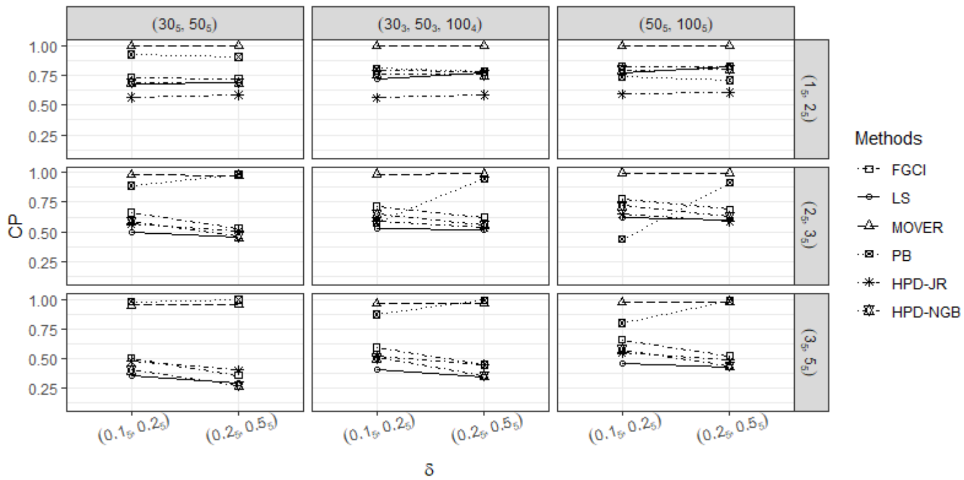

Figure 3. CP performances of 95%CIs for ϑ: 10 sample cases

AN EMPIRICAL APPLICATION

Daily rainfall data obtained from the Thai Meteorological Department (TMD) were divided into the northern, northeastern, central, and eastern regions, while the southern region was a combination of the data from the southeastern and southwestern shores. The daily rainfall amount was recorded on August 5, 2019. This date is in the middle of the rainy season (mid-May to mid-October) when rice farming is conducted in Thailand. Entries with rainfall of less than 0.1 mm were considered as zero records.

Table 5 contains the daily rainfall records for the five regions, Figure 4 shows histogram plots of rainfall observations, and Figure 5 reveals normal Q-Q plots of the log-positive rainfall data. It can be seen that the data for all of the regions contained zero observations. After that, the fitted distribution of the positive observations was checked using the Akaike information criterion (AIC), as reported in Table 6. It can be concluded that the rainfall data in all of the regions followed a delta-lognormal distribution. The summary statistics are reported in Table 7. In the approximation of the daily rainfall in the five regions, the estimated common mean was 4.4506 mm/day. The 95%CIs of the common rainfall mean were computed and reported in Table 8. Under the rain criteria issued by the TMD (Department, 2018), it can be interpreted that the daily rainfall in Thailand is light (0.1–10.0 mm). These results confirm the simulation results for $k = 5$ in the previous section.

DISCUSSION

It can be seen that for MOVER and PB developed from the studies of Krishnamoorthy and Oral (2015) and Malekzadeh and Kharrati-Kopaei (2019), respectively, the simulation results are similar to both of these studies provided that the zero observations are omitted. CIs for the common mean have been investigated in both normal and lognormal distributions (Fairweather, 1972; Jordan and Krishnamoorthy, 1996; Krishnamoorthy and Mathew, 2003; Lin and Lee, 2005; Tian and Wu, 2007; Krishnamoorthy and Oral, 2015). However, the common mean of delta-lognormal populations is especially of interest because it can be used to fit the data from real-world situations such as investigating medical costs (Zou et al., 2009; Tierney et al., 2003; Tian, 2005), analyzing airborne contaminants (Owen and DeRouen, 1980; Tian, 2005) and measuring fish abundance (Fletcher, 2008; Wu and Hsieh, 2014). Our findings show that some of the methods studied had CPs that were too low or too high for large sample cases, a shortcoming that should be addressed in future work.

CONCLUSIONS

The objective of this study was to propose CIs for the common mean of several delta-lognormal distributions using FGCI, LS, MOVER, PB, HPD-JR, and HPD-NGB. The CP and AL as performance measures of the methods were assessed via Monte Carlo simulation. The findings confirm that for the small sample case $(k = 2)$, FGCI and HPD-NGB are the recommended methods in different situations:

Table 5. Daily rainfall data in five Thailand's regions

Northern		Northeastern							Central		Eastern	Southern			
3	0	3	0	0	49.5	0	0	0	2.9	3.2	0	4.1	0	0	2.7
2.6	5	0	40	1.5	10.5	0	0	0	0.2	0	3.2	0	0	0	0
1	23.8	0	3.5	18.5	60.4	4	0	11	0.3	0	10.4	11.5	3.5	0	0
3.6	16	0	0	42	12.7	0	0	0	2.5	4.7	1.1	2.5	13.6	0	0
0	11.5	0	12	9.1	6.8	0	20.3	0	0.4	19.3	0.2	9.7	0	0.2	0
13.2	1.2	0	15	6	69.3	0	0	0	0.4	3.1	4.3	10.4	0	0	0
22.4	10.3	0	0	7.5	36.5	0	2.4	0.3	1.1	2.9	0	9.6	0	0	0
1.4	1.7	0	1.5	0	8.6	0	0	1	0	5.7	0	19	0	0	0
18.3	5.5	0	0.7	6.3	0	0	0	0	1.3	0.9	0	8.3	0	0	0
0	7.3	0	0	0	0	0	0	0	0.1	0	0	0	4.8	0	6.2
15.5	24.3	1.7	3	0.4	0	0	0	0	2.9	0	0.2	0	0	0	0
0	27.2	2.3	0	0	3.8	0	0	0	0	2.6	0.1	0	0	0	0
0	12.6	0.5	0	0	0	0	3.2	0	1	17	62.8	0	0	0	6.1
0	22.7	3.9	0	0	0	0	0	0	4.7	0	36.7	17.8	0	0	0
9.8	0	6.9	29.4	1.8	0	0	0	0	0.5	3.5	15.6	12.3	0	0	0
24.3	2.6	2.2	48	0	0	0	0	0	5	0	50	2.5	0	0	0
24.6	0	3.2	0	0	0	6	0	0	2.5	0	35.5	0	0	0	0.3
8.8	3.2	5.3	70.8	14.3	0	0	0	0	0	0	35	0.9	0	0	0
0	2.6	11	3.5	0	0	0	0	0	0	5.1	5.9	0	0	0	0
19.8	2	0.6	14.2	0	0	0	4.8	0	0	60.4	0	2.6	0	0	0
5	8	0	7	0	0	2.3	0	0	0	6.9	0	0	0	0	0
12.3	1.9	1	0	0	21.5	0	0	0	6.6	3	3	0	0	0	0
8.1	0.8	2.4	0	0	2.5	1	0	0	0	15.1	60.4	2	0	0	0
4.8	2.2	13.2	0	0	0	0	0	0	9.5	6	60	0	0	0	0
5.8	6.5	0.4	0	0	13	0	0		5.1	13.4	76	0	0	0	0
17	0	0	10.8	0	26.2	0	0		12.5	6.2	79.7	0	0	0	0
25.1	2.2	1.3	0	10.1	2.2	4.6	5.4		0		65.7	3.5	0	0	
8.3	0	10	6.3	0	3	0	0		0		108	0	0	36.1	
22.9	4.3	2.5	0	4.8	10.5	10	0		3.2		10.5	0	0	41.8	
26.9	0.2	4.6	4	0	0	0	12		0			0	0	30	
0	0	0	19.3	0	0	9.5	0		2.2			0	0	0	

Source: Thai Meteorological Department
https://www.tmd.go.th/services/weekly_report.php

Table 6. AIC results of daily rainfall records in five Thailand's regions

Regions	AIC				
	Cauchy	Logistic	Lognormal	Normal	T-distribution
Northern	373.1958	357.3122	**336.8724**	353.7757	354.3055
Northeastern	600.9473	642.1779	**543.9619**	667.2334	664.6152
Central	240.0227	266.4162	**220.8503**	293.9151	283.2302
Eastern	229.8995	220.2523	**202.8394**	218.7240	219.1471
Southern	194.9368	197.5586	**178.5587**	201.1654	200.1388

Table 7. The summary statistics

Regions	Estimated paramters				
	n_i	$\hat{\mu}_i$	$\hat{\sigma}_i^2$	$\hat{\delta}_i$	$\hat{\vartheta}_i$
Northern	62	1.866	1.277	0.210	9.472
Northeastern	210	1.734	1.578	0.619	4.668
Central	57	1.085	1.784	0.316	4.741
Eastern	29	2.366	4.545	0.241	59.391
Southern	119	1.684	1.730	0.782	2.639

Table 8. 95%CIs of common rainfall mean in five Thailand's regions

Regions	95%CIs for ϑ		Langths
	Lower	Upper	
FGCI	2.5545	6.3342	3.7798
LS	3.2166	5.6846	2.4681
MOVER	2.7216	9.0296	6.3080
PB	5.8876	11.4965	5.6089
HPD-JR	3.5216	7.8533	4.3317
HPD-NGB	2.4969	6.0904	3.5935

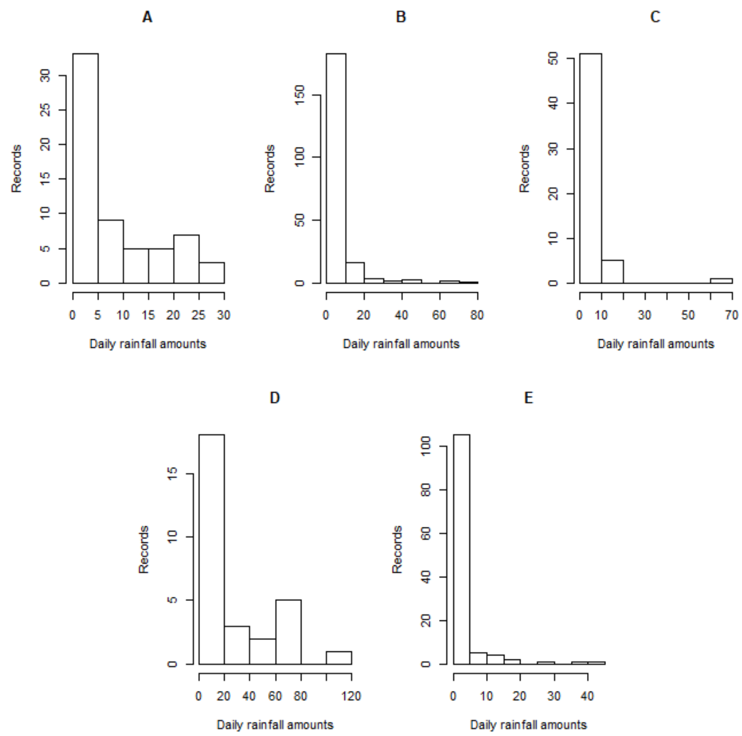

Figure 4. Histogram plots of daily rainfall data in five Thailand's regions: (A) northern (B) northeastern (C) central (D) eastern (E) southern

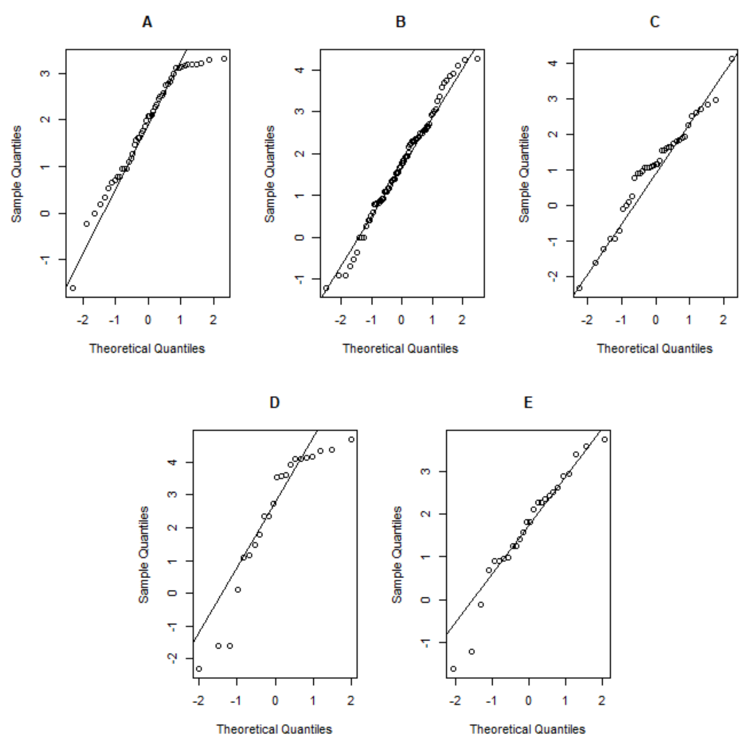

Figure 5. Normal O-Q plots of log-positive daily rainfall data in five Thailand's regions: (A) northern (B) northeastern (C) central (D) eastern (E) southern

FGCI (small-to-moderate sample size, as well as for large σ_i^2 with a moderate-to-large sample size) and HPD-NGB (small σ_i^2 with a moderate-to-large sample size). For large sample cases $(k = 5, 10)$, MOVER (small δ_i and σ_i^2) and PB (large δ_i and σ_i^2) performed the best.

REFERENCES

Aitchison, J. (1955). On the distribution of a positive random variable having a discrete probability mass at the origin. *Journal of the American Statistical Association*, 50(271):901–908.

Aitchison, J. and Brown, J. A. (1963). *The lognormal distribution: with special reference to its uses in economics*. Cambridge University Press.

Box, G. E. P. and Tiao, G. C. (1973). *Bayesian Inference in Statistical Analysis*. New York: Wiley Classics.

Casella, G. and Berger, R. L. (2002). *Statistical Inference (2nd ed.)*. Duxbury, Pacifc Grove, CA.

Department, T. M. (2018). Meteorological academic, knowledge and document. Available from https://www.tmd.go.th/info/info.php?FileID=68.

Fairweather, W. R. (1972). A method of obtaining an exact confidence interval for the common mean of several normal populations. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 21(3):229–233.

Fisher, R. A. (1930). Inverse Probability. *Mathematical Proceedings of the Cambridge Philosophical Society*, 26(4):528–535.

Fletcher, D. (2008). Confidence intervals for the mean of the delta-lognormal distribution. *Environmental and Ecological Statistics*, 15(2):175–189.

Graybill, F. A. and Deal, R. B. (1959). Combining Unbiased Estimators. *Biometrics*, 15(4):543.

Hannig, J. (2009). On generalized fiducial inference. *Statistica Sinica*, 19(2):491–544.

Hannig, J., Iyer, H., and Patterson, P. (2006). Fiducial generalized confidence intervals. *Journal of the American Statistical Association*, 101(473):254–269.

Harvey, J. and van der Merwe, A. J. (2012). Bayesian confidence intervals for means and variances of lognormal and bivariate lognormal distributions. *Journal of Statistical Planning and Inference*, 142(6):1294–1309.

Hasan, M. S. and Krishnamoorthy, K. (2018). Confidence intervals for the mean and a percentile based on zero-inflated lognormal data. *Journal of Statistical Computation and Simulation*, 88(8):1499–1514.

Jordan, S. M. and Krishnamoorthy, K. (1996). Exact confidence intervals for the common mean of several normal populations. *Biometrics*, 52(1):77–86.

Krishnamoorthy, K. and Mathew, T. (2003). Inferences on the means of lognormal distributions using generalized p-values and generalized confidence intervals. *Journal of Statistical Planning and Inference*, 115(1):103–121.

Krishnamoorthy, K. and Oral, E. (2015). Standardized likelihood ratio test for comparing several log-normal means and confidence interval for the common mean. *Statistical Methods in Medical Research*, 26(6):2919–2937.

Kunio, S. (1983). Variances of UMVU Estimators for Means and Variances After Using a Normalizing Transformation. *Communications in Statistics - Theory and Methods*, 12(9):975–985.

Li, X., Zhou, X., and Tian, L. (2013). Interval estimation for the mean of lognormal data with excess zeros. *Statistics & Probability Letters*, 83(11):2447–2453.

Lin, S.-H. and Lee, J. C. (2005). Generalized inferences on the common mean of several normal populations. *Journal of Statistical Planning and Inference*, 134(2):568–582.

Lin, S. H. and Wang, R. S. (2013). Modified method on the means for several log-normal distributions. *Journal of Applied Statistics*, 40(1):194–208.

Longford, N. T. (2009). Inference with the lognormal distribution. *Journal of Statistical Planning and Inference*, 139(7):2329–2340.

Malekzadeh, A. and Kharrati-Kopaei, M. (2019). Inferences on the common mean of several heterogeneous log-normal distributions. *Journal of Applied Statistics*, 46(6):1066–1083.

Maneerat, P., Niwitpong, S.-a., and Niwitpong, S. (2020). A Bayesian approach to construct confidence intervals for comparing the rainfall dispersion in Thailand. *Peerj*, 8:e8502.

Miroshnikov, A., Wei, Z., and Conlon, E. M. (2015). Parallel Markov chain Monte Carlo for non-Gaussian posterior distributions. *Stat*, 4:304–319.

Owen, W. J. and DeRouen, T. A. (1980). Estimation of the mean for lognormal data containing zeroes

and left- censored values, with applications to the measurement of worker exposure to air contaminants. *Biometrics*, 36(4):707–719.
Shimizu, K. and Iwase, K. (1981). Uniformly minimum variance unbiased estimation in lognormal and related distributions. *Communications in Statistics - Theory and Methods*, 10(11):1127–1147.
Supasod, W. (2006). Development of Water Resources for Agriculture. Technical report. Available from: http://www.rid.go.th/thaicid/_5_article/2549/03Water_Agri.pdf.
Thai Meteorological Department (2014). Mean annual rainfall in Thailand over 30 year. Technical report. Available from: https://www.tmd.go.th/en/climate.php?FileID=7.
Tian, L. (2005). Inferences on the mean of zero-inflated lognormal data: the generalized variable approach. *Statistics in Medicine*, 24(20):3223–3232.
Tian, L. and Wu, J. (2007). Inferences on the Common Mean of Several Log-Normal Populations: The Generalized Variable Approach. *Biometrical Journal*, 49(6):944–951.
Tierney, W. M., Overhage, J. M., Murray, M. D., Harris, L. E., Zhou, X.-H., Eckert, G. J., Smith, F. E., Nienaber, N., McDonald, C. J., and Wolinsky, F. D. (2003). Effects of computerized guidelines for managing heart disease in primary care. *Journal of General Internal Medicine*, 18(12):967–976.
Wu, W.-H. and Hsieh, H.-N. (2014). Generalized confidence interval estimation for the mean of delta-lognormal distribution: an application to New Zealand trawl survey data. *Journal of Applied Statistics*, 41(7):1471–1485.
Zhou, X.-H., Gao, S., and Hui, S. L. (1997). Methods for Comparing the Means of Two Independent Log-Normal Samples. *Biometrics*, 53(3):1129–1135.
Zhou, X. H. and Tu, W. (2000). Confidence intervals for the mean of diagnostic test charge data containing zeros. *Biometrics*, 56(4):1118–1125.
Zou, G. Y., Taleban, J., and Huo, C. Y. (2009). Confidence interval estimation for lognormal data with application to health economics. *Computational Statistics & Data Analysis*, 53(11):3755–3764.